Baryon wave function in large-N_c QCD: Universality, nonlinear evolution equation and asymptotic limit

P.V. Pobylitsa
Institute for Theoretical Physics II,
Ruhr University Bochum, D-44780 Bochum, Germany
and
Petersburg Nuclear Physics Institute,
Gatchina, St. Petersburg, 188300, Russia

The $1/N_c$ expansion is formulated for the baryon wave function in terms of a specially constructed generating functional. The leading order of this $1/N_c$ expansion is universal for all low-lying baryons [including the $O(N_c^{-1})$ and $O(N_c^0)$ excited resonances] and for baryon-meson scattering states. A nonlinear evolution equation of Hamilton-Jacobi type is derived for the generating functional describing the baryon distribution amplitude in the large-N_c limit. In the asymptotic regime this nonlinear equation is solved analytically. The anomalous dimensions of the leading-twist baryon operators diagonalizing the evolution are computed analytically up to the next-to-leading order of the $1/N_c$ expansion.

PACS numbers: 11.15.Pg, 11.10.Hi, 12.38.Lg, 14.20.-c

I. INTRODUCTION

The limit of the large number of colors $N_c \to \infty$ and the $1/N_c$ expansion in QCD are powerful methods of theoretical analysis which provide us not only with a qualitative understanding of nonperturbative phenomena of QCD but also sometimes lead to interesting quantitative results. The history of the applications of the $1/N_c$ expansion in QCD is almost as old [1, 2, 3, 4] as QCD itself and the roots of this method in statistical and many-body physics are even deeper.

In spite of this long history some quantities important for the phenomenological applications of QCD have for a long time remained out of the scope of the large-N_c community. In this paper we construct the $1/N_c$ expansion for the baryon wave function. The light-cone version of the baryon wave function (distribution amplitude) plays a crucial role in the QCD analysis of hard exclusive phenomena [5, 6, 7, 8].

Apart from the importance for practical applications, the baryon wave function is a quantity with a very interesting structure of the $1/N_c$ expansion. The traditional methods of the $1/N_c$ expansion usually deal with quantities that have a *power* large-N_c behavior. For a vast amount of baryon properties (masses, mass splittings, magnetic moments, form factors, structure functions, etc.), the large-N_c behavior can be described by a power counting of N_c. On the contrary, the large-N_c analysis of the baryon wave function requires the development of methods that can be applied to quantities which depend on N_c *exponentially.*

The main part of this paper is devoted to the light-cone case which is especially interesting for practical applications. However, for simplicity we prefer to start the analysis of the large-N_c behavior from the "covariant quark wave function" $\hat{\Psi}^B$ of a baryon $|B\rangle$,

$$\hat{\Psi}^B_{(f_1 s_1)(f_2 s_2)\dots(f_{N_c} s_{N_c})}(z_1, z_2, \dots, z_{N_c}) \equiv \frac{1}{N_c!}\varepsilon_{c_1\dots c_{N_c}} \langle 0| \mathcal{T}\left[q_{c_1 f_1 s_1}(z_1) q_{c_2 f_2 s_2}(z_2)\dots q_{c_{N_c} f_{N_c} s_{N_c}}(z_{N_c})\right]|B\rangle \,. \tag{1.1}$$

Here $q_{cfs}(z)$ is the quark field taken at the space-time point z, and c, f, s are color, flavor and spin indices, respectively. The Levi-Civita tensor $\varepsilon_{c_1\dots c_{N_c}}$ is totally antisymmetric in color indices. We use the notation $\hat{\Psi}$ with a hat in order to distinguish the covariant wave function (1.1) from the light-cone case, which is the main subject of the paper. Because of the time ordering $\mathcal{T}$ and Fermi statistics of quarks, the function $\hat{\Psi}^B_{(f_1 s_1)\dots(f_{N_c} s_{N_c})}(z_1, \dots, z_{N_c})$ is symmetric with respect to permutations of (z_k, f_k, s_k). Special care should be taken about the local gauge invariance. We assume an insertion of Wilson lines on the RHS of Eq. (1.1) in such a way that function $\hat{\Psi}^B$ becomes gauge invariant but still remains symmetric with respect to permutations of (z_k, f_k, s_k) (for example, we can choose straight Wilson lines connecting each quark field with the "center of mass" of points $z_1, z_2, \dots, z_{N_c}$ and antisymmetrize the N_c color indices at this center point).

Our aim is to understand the large-N_c behavior of the baryon wave function (1.1). But if we try to apply the $1/N_c$ expansion to the function (1.1) directly, then we immediately meet a problem. This function depends on N_c arguments. From the mathematical point of view it hardly makes sense to speak about the large-N_c asymptotic behavior of a function depending on N_c variables. For a consistent treatment of the large-N_c limit we need some intermediate object which

1) contains the same information as the wave function $\hat{\Psi}^B_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}(z_1, z_2, \ldots, z_{N_c})$,
2) allows to take the large-N_c limit and to construct the $1/N_c$ expansion.
To this aim we introduce the generating functional

$$\hat{\Phi}_B(g) = \sum_{f_k s_k} \int dz_1 \int dz_2 \ldots \int dz_{N_c} g_{f_1 s_1}(z_1) g_{f_2 s_2}(z_2) \ldots g_{f_{N_c} s_{N_c}}(z_{N_c}) \hat{\Psi}^B_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}(z_1, z_2, \ldots, z_{N_c}) \quad (1.2)$$

depending on an arbitrary "source" function $g_{fs}(z)$. Since the function $\hat{\Psi}^B$ is totally symmetric in permutations of (z_k, f_k, s_k), at finite N_c we can restore the wave function $\hat{\Psi}^B$ if we know the functional $\hat{\Phi}_B(g)$:

$$\hat{\Psi}^B_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}(z_1, z_2, \ldots, z_{N_c}) = \frac{1}{N_c!} \left[\prod_{k=1}^{N_c} \frac{\delta}{\delta g_{f_k s_k}(z_k)} \right] \hat{\Phi}_B(g) \, . \quad (1.3)$$

What can we say about the large-N_c asymptotic behavior of the functional $\hat{\Phi}_B(g)$? In principle, it is a dynamic problem and without solving large-N_c QCD we cannot give an absolutely reliable answer. Nevertheless on general grounds it is natural to expect the behavior of the type

$$\hat{\Phi}_B(g) = \left\{ N_c^{\nu_B} \hat{A}_B(g) \exp\left[N_c \hat{W}(g) \right] \right\} \left[1 + O(N_c^{-1}) \right] \, . \quad (1.4)$$

Here the exponential behavior is controlled by the functional $\hat{W}(g)$. The pre-exponential factor $\hat{A}_B(g)$ may be accompanied by a power term $N_c^{\nu_B}$. In this paper we present a number of arguments in favor of the large-N_c behavior (1.4). We shall also establish some interesting properties of functionals $\hat{W}(g)$ and $\hat{A}_B(g)$. In particular, we shall show that the functional $\hat{W}(g)$ is universal for all low-lying baryons (see Sec. III for details).

II. MAIN RESULTS

A. Notation

In this paper we deal with the $1/N_c$ expansion for similar but different versions of the baryon wave function. In all cases we introduce generating functionals analogous to $\hat{\Phi}_B(g)$ (1.2) and describe their large-N_c asymptotic behavior in terms of functionals similar to $\hat{W}(g)$ and $\hat{A}_B(g)$ in Eq. (1.4). In order to avoid confusion we use a slightly different notation for various cases:

- covariant baryon wave function $\hat{\Psi}^B$ (1.1) and the associated functionals $\hat{\Phi}_B(g)$, $\hat{W}(g)$, $\hat{A}_B(g)$,
- baryon wave function ψ^B (4.2) in a toy quark model and functionals $\phi_B(g)$, $w(g)$, $a_B(g)$,
- baryon light-cone wave function (distribution amplitude) Ψ^B (5.5) and functionals $\Phi_B(g)$, $W(g)$, $A_B(g)$,
- asymptotic baryon distribution amplitude and other wave functions $\tilde{\Psi}^B$ diagonalizing the evolution kernel (9.5), and functionals $\tilde{\Phi}_B(g)$, $W_0(g)$, $\tilde{A}_B(g)$.

On the other hand, the general structure of the $1/N_c$ expansion in all these cases is almost identical. Therefore in the rest of *this* section describing the main results of the paper, we use the same notation $\Phi_B(g)$, $W(g)$, $A_B(g)$ for all above cases. The precise meaning of these functionals will be obvious from the context.

B. General structure of the $1/N_c$ expansion and universality of the functional $W(g)$

One of the aims of this paper is to check the consistency of the large-N_c structure (1.4) of the functional $\Phi_B(g)$. The traditional machinery of the large-N_c analysis includes several approaches:

- equivalence of the large-N_c counting in QCD and in various models imitating QCD [1, 2, 3, 4, 9, 10, 11] (nonrelativistic quark model [12, 13, 14], Skyrme model [15, 16, 17], chiral quark-soliton model [18, 19, 20], etc.),
- direct counting of N_c orders in perturbative Feynman diagrams [1, 2, 3, 4],

- methods based on the spin-flavor symmetry and consistency condition [9, 10, 21, 22, 23].

In Sec. IV we compute the functional $\Phi_B(g)$ in a toy quark model and show explicitly that the large-N_c behavior of this functional has the structure (1.4). Using this toy model, we describe the saddle point method which will be later applied to the analysis of the large-N_c behavior of the baryon distribution amplitude in QCD.

Now let us turn to the role of perturbative QCD in the analysis of the large-N_c behavior of the baryon wave function. *Perturbative* Feynman diagrams allow us to estimate the N_c order of various *nonperturbative* quantities even if the perturbation theory does not work for these quantities. Unfortunately this method cannot be applied to $\Phi_B(g)$ directly. Indeed the functional $\Phi_B(g)$ depends on N_c exponentially so that arbitrarily high orders of N_c can be met in perturbative Feynman diagrams. Nevertheless perturbative QCD provides a very important tool which allows us to check the large-N_c structure of $\Phi_B(g)$ described by Eq. (1.4) — we mean the evolution equation for the baryon distribution amplitude. The main part of this paper is devoted to the evolution of the baryon distribution amplitudes at large N_c.

We stress that in this paper we are not interested in the physical applications of the evolution of the baryon distribution amplitude. The $O(N_c)$ growth of the nucleon mass at large N_c changes the physical picture of hard processes in large-N_c QCD. In fact, in practical applications there is no need in the large-N_c approximation for the evolution.

Instead, we would like to use the evolution equation as a consistency test for the large-N_c structure (1.4). In some sense the evolution equation plays the same role in the analysis of the *exponential* large-N_c behavior as the perturbative Feynman diagrams in the analysis of quantities with the *power* large-N_c asymptotic behavior. In physical applications to hard processes the evolution equations are used for the exponentiation of large logarithms of the hard momentum $\log Q$. In the large-N_c analysis of the baryon distribution amplitude, the evolution equation is helpful due to its ability to exponentiate N_c.

An important property of the $1/N_c$ expansion is the universality of the functional $W(g)$: this functional is the same for all low-lying baryons [including $O(N_c^{-1})$ and $O(N_c^0)$ excited resonances]. Moreover, the same functional $W(g)$ can be used for the quark wave functions of baryon-meson scattering states. This universality property is described in detail in Sec. III. All methods used for the analysis of the large-N_c behavior in this paper (toy quark model, evolution equations, asymptotic limit) confirm the universality of the functional $W(g)$.

In contrast to the $W(g)$, the functional $A_B(g)$ appearing as a pre-exponential factor in Eq. (1.4) depends on the baryon (baryon-meson state) B. We find interesting factorization properties of the functional $A_B(g)$ which are briefly described in Sections III D and X. The complete analysis of these factorization properties would require the techniques based on the large-N_c contracted $SU(2N_F)$ spin-flavor symmetry [9, 10, 21, 22, 23]. Here we do not touch upon this method, leaving it for a separate paper [24].

C. Nonlinear evolution equation and its Hamilton-Jacobi structure

In Sec. V A we introduce the light-cone version of the functional $\Phi_B(g)$ describing the baryon distribution amplitude and define the corresponding functionals $W(g)$ and $A_B(g)$ using the large-N_c representation (1.4). In Sec. VI we derive the evolution equations for $W(g)$ and $A_B(g)$. The resulting evolution equation (6.13) for the functional $W(g)$ is nonlinear. This nonlinear evolution equation has the Hamilton-Jacobi form (Sec. VI D). The asymptotic limit of the large normalization scale $\mu \to \infty$ is studied in Sections VII, VIII. In Sec. IX we solve the problem of the diagonalization of the evolution of the baryon distribution amplitude at large N_c and compute the corresponding anomalous dimensions in two orders of the $1/N_c$ expansion.

D. Conformal symmetry and integrability

An essential part of this paper is devoted to the evolution equation and to the problem of the diagonalization of anomalous dimensions of baryon operators at large N_c. At $N_c = 3$ this problem has attracted much attention [25, 26, 27, 28, 29, 30] and was studied with various methods ranging from straightforward numerical attacks to the discovery of the integrability of the evolution equation for baryons with helicity 3/2 [31, 32].

The conformal symmetry of the evolution equation is known to play an important role in this problem [6, 27, 33, 34, 35, 36]. It is well known that the large-N_c limit is often helpful for solving problems inspired by high-energy QCD [37, 38] where the dynamics is determined by the *gluon* sector. In these problems the large-N_c limit leads to the "spin chains" with the interaction only between the next neighbors [39]. The case of the *quark* distribution amplitudes of baryons is different. In this case we have equally important pair interactions between all N_c quarks. We derive a variational equation for the asymptotic limit of the functional $W(g)$ and solve this equation analytically in Sec. VIII.

Technically the solution is found in terms of a successfully guessed ansatz but the role of the conformal symmetry standing behind this ansatz is rather visible. Another problem solved analytically in this paper is the diagonalization of the anomalous dimensions of baryon operators of leading twist in two orders of the $1/N_c$ expansion [$O(N_c)$ and $O(N_c^0)$]. The calculation of these anomalous dimensions in the next-to-leading $O(N_c^0)$ order is based on the solution of rather nontrivial functional equations. The analytical solutions of these equations are found in Sec. IX C.

E. Phenomenological applications

On the phenomenological side one is certainly interested in the calculation of the functionals $W(g)$ and $A_B(g)$ appearing in the $1/N_c$ expansion (1.4). This challenging problem belongs to the same class of difficulty as solving large-N_c QCD. Although we do not know the functionals $W(g)$ and $A_B(g)$, in this paper we establish several properties which may be of interest for the phenomenological applications. One of them is the universality of the functional $W(g)$ which allows a unified description of different baryons and meson-baryon scattering states.

The large-N_c arguments are sometimes used [40, 41] for the justification of various approximations in phenomenological models of baryons. Our results impose certain theoretical restrictions on these models. As was explained above, it is probably impossible to construct a systematic $1/N_c$ expansion for the baryon wave function directly, without using auxiliary objects like the generating functional (1.2). Therefore the models compatible with the $1/N_c$ expansion can access only the weighted integrals of the baryon wave function rather than the wave function itself. One should keep in mind that the one-to-one correspondence (1.2), (1.3) between the wave function Ψ^B and the functional $\Phi_B(g)$ holds only at finite N_c. We cannot compute the wave function Ψ^B at finite N_c by applying the variational derivatives of Eq. (1.3) to the large-N_c asymptotic expression (1.4) for $\Phi_B(g)$.

Another theoretical restriction imposed on the phenomenological analysis by the systematic $1/N_c$ expansion is the necessity to work with the N_c-point baryon wave function [via the generating functional $\Phi_B(g)$]. The arguments of the $1/N_c$ expansion cannot be used in models explicitly dealing with baryons "made of three quarks". Although one may think that working with the 3-point baryon wave function is equivalent to a partial resummation of higher $1/N_c$ corrections, the restriction to the 3-point baryon wave functions is incompatible with the systematic $1/N_c$-expansion.

Obviously the universality of the exponential large-N_c behavior for all low-lying baryons is not sufficient for serious practical applications. One also needs a good control of the pre-exponential non-universal functionals $A_B(g)$. The factorizable structure of these functionals is studied in detail in Ref. [24] where an interesting relation connecting the cases of nucleon and Δ resonance is derived.

III. UNIVERSALITY OF THE GENERATING FUNCTIONAL $\hat{W}(g)$

A. Universality of $\hat{W}(g)$ for baryons

As was already mentioned, the functional $\hat{W}(g)$ appearing in the large-N_c decomposition (1.4) is universal for all low-lying baryons. This property is rather general: it also holds in the case of the light-cone distribution amplitude which is the main subject of the paper and in the exactly solvable toy quark model that will be considered in Sec. IV. In order to be specific, we describe this universality property for the case of the covariant baryon wave function (1.1) keeping in mind that all statements made in this section are also valid in the light-cone case.

First we must specify the baryons for which the universality of the functional $\hat{W}(g)$ is expected. According to the standard picture of baryons in large-N_c QCD with $N_f = 2$ flavors, the lowest baryons have equal spin J and isospin T [16, 42]:

$$T = J = \begin{cases} \frac{1}{2}, \frac{3}{2}, \ldots & \text{for odd } N_c\,, \\ 0, 1, 2, \ldots & \text{for even } N_c\,. \end{cases} \tag{3.1}$$

The masses of these baryons have a $1/N_c$ suppressed splitting:

$$M_{T=J} = d_1 N_c + d_0 + N_c^{-1}\left[d_{-1}^{(1)} + d_{-1}^{(2)} J(J+1)\right] + O(N_c^{-2})\,. \tag{3.2}$$

These baryons belong to the same representation of the spin-flavor symmetry group [9, 10, 21, 22, 23] which becomes asymptotically exact at large N_c. As a consequence, the functional $\hat{W}(g)$ introduced in Eq. (1.4) is the same for all low-lying baryons (3.1). The dependence on the type of baryons B appears only in the pre-exponential factor $\hat{A}_B(g)$.

Moreover, at large N_c one can also consider higher baryon resonances with the excitation energy $O(N_c^0)$. Their mass spectrum is described by the formula

$$M = d_1 N_c + \left(d_0 + \sum_i \omega_i n_i\right) + O(N_c^{-1})\,, \tag{3.3}$$

where $n_i = 0, 1, 2, \ldots$ are integer numbers, and the ω_i are N_c independent coefficients [we have omitted the higher $O(N_c^{-1})$ correction depending on additional quantum numbers]. At large $N_c \gg n_i$ the functional $\hat{\Phi}(g)$ for these baryons also has the structure (1.4) with the same "universal" functional $\hat{W}(g)$ but with baryon-dependent pre-exponential functionals $\hat{A}_B(g)$.

B. Baryon-meson scattering states

Strictly speaking, the existence of the states (3.3) depends on the mass thresholds which control the decays of excited baryons into lower baryons and mesons. If these decays are possible, then the baryons (3.3) have a parametrically large width $O(N_c^0)$. In this case instead of large-width baryons it is better to work directly with the scattering states containing one baryon and one (or several) meson [43, 44, 45, 46, 47, 48]. We still can use expression (3.3) for the description of the energy of these scattering states identifying parameters ω_i with the energies of single mesons in the rest frame of the $O(N_c)$ heavy baryon.

For these baryon-meson scattering states one can also define the wave function (1.2) with the large-N_c behavior (1.4). This representation (1.4) will contain the same "universal" functional $\hat{W}(g)$ as before (we assume that the number of mesons in the scattering state is kept fixed in the limit of large N_c).

The spectrum of applications of the universality of the functional $\hat{W}(g)$ can be extended. For example, instead of the scattering states with one baryon and on-shell mesons we can consider the matrix elements with additional insertions of off-shell quark-antiquark color-singlet currents

$$J_k(y) = \sum_{c=1}^{N_c} \bar{q}_c(y)\mathcal{S}_k q_c(y)\,, \tag{3.4}$$

where $\mathcal{S}_k$ is some spin-flavor matrix. If the number n of these insertions is kept fixed in the large-N_c limit, then the large-N_c behavior

$$\begin{aligned}
&\frac{1}{N_c!}\varepsilon_{c_1\ldots c_{N_c}} \sum_{f_k s_k} \int dz_1 \int dz_2 \ldots \int dz_{N_c} g_{f_1 s_1}(z_1) g_{f_2 s_2}(z_2) \ldots g_{f_{N_c} s_{N_c}}(z_{N_c}) \\
&\times \langle 0|\mathcal{T}\left\{\left[\prod_{k=1}^{n} J_k(y_k)\right] q_{c_1 f_1 s_1}(z_1) q_{c_2 f_2 s_2}(z_2) \ldots q_{c_{N_c} f_{N_c} s_{N_c}}(z_{N_c})\right\}|B\rangle \\
&\qquad = N_c^{\nu_{B,\{J_k\}}} \hat{A}_{B,\{J_k(y_k)\}}(g) \exp\left[N_c \hat{W}(g)\right]
\end{aligned} \tag{3.5}$$

is described by Eq. (1.4) with the same functional $\hat{W}(g)$ as in the case of the baryon wave function.

Let us consider QCD with the exact $SU(2)$ flavor invariance. At finite N_c the functional $\hat{\Phi}_B(g)$ is invariant if we simultaneously perform a flavor rotation of the quark fields and of the baryon state. But due to the universality of the functional $\hat{W}(g)$ the flavor rotation of the baryon state is not needed. Thus we conclude that functional $\hat{W}(g)$ is invariant under flavor rotations of g. Similarly, the universality of $\hat{W}(g)$ for baryons with different polarizations leads to the invariance of $\hat{W}(g)$ with respect to the spacial rotations of g [including both the rotation of the space argument $\mathbf{z}$ and the transformation of the spin index s of $g_{fs}(z^0, \mathbf{z})$].

C. Universality and $SU(N_f)$ symmetry

Note that the universality of $W(g)$ is based on certain assumptions about the large-N_c behavior of the quantum numbers of baryons. For example, in the case of the $SU(N_f)$ flavor symmetry with $N_f = 2$, the functional $\hat{W}(g)$ is universal only for those baryons whose spin J and isospin T do not grow with N_c:

$$I, J = O(N_c^0)\,. \tag{3.6}$$

In the case of $N_f = 3$ we must also keep the strangeness S bounded in the limit $N_c \to \infty$

$$S = O(N_c^0)\,, \tag{3.7}$$

if we want to have the universality of $\hat{W}(g)$.

D. Factorization of the pre-exponential factors

In contrast to the universality of the $\hat{W}(g)$, the functional $\hat{A}_{B,\{J_k(y_k)\}}(g)$ depends both on the type of the baryon and on the currents $J_k(y_k)$. An important property of the functional $\hat{A}_{B,\{J_k(y_k)\}}(g)$ is its factorization. In an oversimplified form this factorization is

$$\hat{A}_{B,\{J_k(y_k)\}}(g) = \hat{A}_0(g) \prod_i \left[\hat{\xi}_i(g)\right]^{n_i} \prod_k \left[\hat{\eta}_{J_k}(g)\right]\,, \tag{3.8}$$

where the functional $\hat{A}_0(g)$ corresponds to the wave function of the lowest baryon, the factors $\hat{\xi}_i(g)$ are associated with the elementary ω_i excitations in Eq. (3.3), and the factors $\hat{\eta}_{J_k}(g)$ correspond to the insertions of currents J_k in Eq. (3.5). The precise expression for $\hat{A}_{B,\{J_k(y_k)\}}(g)$ must also contain functional factors which come from the zero modes corresponding to the spin-flavor rotations. The role of zero modes is discussed in Sec. X C and in Ref. [24].

IV. TOY QUARK MODEL

A. Model

We want to compute the analog $\phi_B(g)$ of the functional $\Phi_B(g)$ (1.2) in a simple toy model and to show explicitly how the large-N_c asymptotic behavior (1.4) appears. This can be done in two ways. First we compute the functional $\phi_B(g)$ exactly and take the large-N_c limit of this exact result. In Sec. IV C we describe another approach based on the saddle point method. The results obtained in the toy model will also be used in Sec. VIII C where we study the asymptotic solutions of the nonlinear evolution equation.

Our toy quark model deals with baryons made of N_c quarks. We assume that quarks with flavor f, spin s and color c are pinned at one point so that the spacial motion is ignored. We consider the case of $N_f = 2$ flavors.

In this simple model, the spin J of a baryon coincides with its isospin T

$$T = J = \begin{cases} \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N_c}{2} & \text{for odd}\, N_c\,, \\ 0, 1, 2, \ldots, \frac{N_c}{2} & \text{for even}\, N_c\,, \end{cases} \tag{4.1}$$

and the quark spin-flavor wave function of a baryon can be written in the form

$$\psi^{TT_3J_3}_{(f_1s_1)(f_2s_2)\ldots(f_{N_c}s_{N_c})} = c_{TN_c} \int dR D^T_{J_3T_3}(R^{-1}) \prod_{k=1}^{N_c} R_{f_ks_k}\,. \tag{4.2}$$

Here $D^j_{mm'}(R)$ are Wigner functions for the group $SU(2)$. The integral runs over the $SU(2)$ matrices R with the Haar measure normalized by the condition

$$\int dR = 1\,. \tag{4.3}$$

The normalization coefficient c_{TN_c} is chosen in Eq. (4.2) so that

$$\sum_{f_ks_k} \left|\psi^{TT_3J_3}_{(f_1s_1)(f_2s_2)\ldots(f_{N_c}s_{N_c})}\right|^2 = 1\,. \tag{4.4}$$

This normalization constant is computed in Appendix A [see Eq. (A16)]

$$c_{TN_c} = \sqrt{\frac{2T+1}{N_c!}\left(\frac{N_c}{2}+T+1\right)!\left(\frac{N_c}{2}-T\right)!}\,. \tag{4.5}$$

B. Generating function $\phi_{TT_3J_3}(g)$

By analogy with Eq. (1.2) we introduce the generating function

$$\phi_{TT_3J_3}(g) = \sum_{f_k s_k} g_{f_1 s_1} \ldots g_{f_{N_c} s_{N_c}} \psi^{TT_3J_3}_{(f_1 s_1)\ldots(f_{N_c} s_{N_c})}$$
$$= c_{TN_c} \int dR D^T_{J_3T_3}(R^{-1}) \left[\mathrm{Tr}\left(R g^{\mathrm{tr}}\right)\right]^{N_c} , \tag{4.6}$$

where g_{fs} is an arbitrary matrix. The superscript tr in g^{tr} stands for the matrix transposition.

In Appendix A we compute $\phi(g)$ at finite N_c [see Eq. (A25)]

$$\phi_{TT_3J_3}(g) = \frac{2T+1}{c_{TN_c}} (\det g)^{N_c/2} D^T_{T_3J_3}\left[\frac{g}{(\det g)^{1/2}}\right] . \tag{4.7}$$

Here $g\,(\det g)^{-1/2}$ is an $SL(2,C)$ matrix. Therefore the Wigner function $D^T_{T_3J_3}$ should be understood in the sense of the complexification of $SU(2)$ to $SL(2,C)$.

At large N_c we find from Eq. (4.5)

$$c_{TN_c} = 2^{-N_c/2}\sqrt{2T+1}\left(\pi N_c^3/8\right)^{1/4} \quad (N_c \gg T) . \tag{4.8}$$

The large-N_c asymptotic behavior of $\phi_{TT_3J_3}(g)$ has the standard form (1.4)

$$\phi_{TT_3J_3}(g) = \left\{N_c^{\nu_B} a_{TT_3J_3}(g) \exp\left[N_c w(g)\right]\right\}\left[1 + O(N_c^{-1})\right] \tag{4.9}$$

with

$$w(g) = \frac{1}{2}\ln\left(2\det g\right) , \tag{4.10}$$

$$\nu_B = -3/4 , \tag{4.11}$$

$$a_{TT_3J_3}(g) = \left(\frac{8}{\pi}\right)^{1/4}\sqrt{2T+1} D^T_{T_3J_3}\left[\frac{g}{(\det g)^{1/2}}\right] . \tag{4.12}$$

We see from Eq. (4.10) that function $w(g)$ is independent of $T = J, T_3, J_3$. This is a manifestation of the "universality" property of $w(g)$ which was discussed in Sec. III.

C. Saddle point method

Although the above results (4.10) and (4.12) for $w(g)$ and $a_{TT_3J_3}(g)$ can be read directly from the exact finite-N_c expression (4.7), it is instructive to derive these results using the large-N_c limit from the very beginning. At large N_c we can apply the saddle point method to the calculation of the integral on the RHS of Eq. (4.6). Indeed, the integrand contains the factor $\left[\mathrm{Tr}\left(R g^{\mathrm{tr}}\right)\right]^{N_c}$ which leads to the saddle point equation

$$\delta_R \mathrm{Tr}\left(R g^{\mathrm{tr}}\right) = 0 . \tag{4.13}$$

The original integral runs in Eq. (4.6) over the $SU(2)$ matrices R. But the saddle point method leads to a deformation of the "integration contour" so that we must look for solutions R and their variation δ_R in the complexification of $SU(2)$, i.e. in $SL(2,C)$. Taking the variation δ_R in the form of an infinitesimal $SL(2,C)$ rotation

$$R' = (1 + i\delta\omega^a \tau^a) R \tag{4.14}$$

with arbitrary complex infinitesimal parameters $\delta\omega^a$, we can rewrite the saddle point equation (4.13) in the form

$$\sum_a \delta\omega^a \mathrm{Tr}\left(\tau^a R g^{\mathrm{tr}}\right) = 0 . \tag{4.15}$$

Thus

$$\mathrm{Tr}\left(\tau^a R g^{\mathrm{tr}}\right) = 0\,. \tag{4.16}$$

We see that

$$\left(R g^{\mathrm{tr}}\right)_{ij} = c\delta_{ij}\,, \tag{4.17}$$

where c is some complex number. Taking the determinant of this equation, we find

$$(\det R)\,(\det g) = c^2\,. \tag{4.18}$$

Since R belongs to $SL(2,C)$, we have

$$\det R = 1\,. \tag{4.19}$$

Hence

$$\det g = c^2\,. \tag{4.20}$$

Now we find two saddle points from Eq. (4.17):

$$R_\pm^{-1} = \pm\frac{g^{\mathrm{tr}}}{(\det g)^{1/2}}\,. \tag{4.21}$$

The calculation of the Jacobian of fluctuations around these saddle points leads to the following general result

$$\int dR f(R)\left[\mathrm{Tr}\left(R g^{\mathrm{tr}}\right)\right]^{N_c} \stackrel{N_c\to\infty}{\longrightarrow} \sum_\pm \frac{2}{\sqrt{2\pi N_c^3}}\left[\mathrm{Tr}\left(R_\pm g^{\mathrm{tr}}\right)\right]^{N_c} f(R_\pm)\,. \tag{4.22}$$

Inserting expression (4.21) for $R_\pm$ and taking $f(R) = D^T_{J_3 T_3}(R^{-1})$, we derive from (4.22)

$$\int dR D^T_{J_3T_3}(R^{-1})\left[\mathrm{Tr}\left(R g^{\mathrm{tr}}\right)\right]^{N_c} \stackrel{N_c\to\infty}{\longrightarrow} \frac{4}{\sqrt{2\pi N_c^3}}\left[2\,(\det g)^{1/2}\right]^{N_c} D^T_{J_3T_3}\left[\frac{g^{\mathrm{tr}}}{(\det g)^{1/2}}\right]. \tag{4.23}$$

Combining this with the asymptotic expression (4.8) for c_{TN_c} and using the property of Wigner functions (A3), we find the saddle point result for the integral (4.6):

$$c_{TN_c}\int dR D^T_{J_3T_3}(R^{-1})\left[\mathrm{Tr}\left(R g^{\mathrm{tr}}\right)\right]^{N_c} \stackrel{N_c\to\infty}{\longrightarrow} \sqrt{2T+1}\left(\frac{8}{\pi N_c^3}\right)^{1/4}(2\det g)^{N_c/2} D^T_{T_3J_3}\left[\frac{g}{(\det g)^{1/2}}\right]. \tag{4.24}$$

We see that we have reproduced the above results (4.10), (4.12) for $w(g)$ and $a_{TT_3J_3}(g)$.

V. GENERATING FUNCTIONAL FOR THE BARYON DISTRIBUTION AMPLITUDE

A. Baryon distribution amplitude

The definition of the baryon distribution amplitude uses an auxiliary light-cone vector n. This vector n can be used to impose light-cone gauge

$$(n\cdot A) = 0\,. \tag{5.1}$$

We are interested in the leading-twist baryon distribution amplitude which is determined in terms of the "good" components of the quark field q,

$$(n\cdot\gamma) q_{cf}(\lambda n)\,. \tag{5.2}$$

In order to separate the "good" components we introduce Dirac spinors u_s associated with the vector n,

$$u_s\otimes\bar u_s = \frac{1}{2}(n\cdot\gamma)(1-2s\gamma_5)\quad\left(s=\pm\frac{1}{2}\right), \tag{5.3}$$

and define the fields

$$\chi_{cfs}(x) = (nP)^{1/2} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \bar{u}_{-s} q_{cf}(\lambda n) \exp\left[i\lambda x(nP)\right] . \tag{5.4}$$

Now we can define the baryon distribution amplitude as the transition matrix element between the vacuum and the baryon B with momentum P:

$$\begin{aligned} &\Psi_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}(x_1, x_2, \ldots, x_{N_c}) \\ &= \frac{1}{N_c!} \varepsilon_{c_1 c_2 \ldots c_{N_c}} \langle 0| \chi_{c_1 f_1 s_1}(x_1) \chi_{c_2 f_2 s_2}(x_2) \ldots \chi_{c_{N_c} f_{N_c} s_{N_c}}(x_{N_c}) |B(P)\rangle . \end{aligned} \tag{5.5}$$

This definition of the distribution amplitude Ψ is independent of the vector n. Usually it is convenient to normalize n by the condition $(nP) = 1$ but we do not impose this constraint. Indeed, at large N_c we have $P = O(N_c)$. We keep n fixed at large N_c, therefore (nP) grows as $O(N_c)$:

$$n = O(N_c^0) , \quad (nP) = O(N_c) . \tag{5.6}$$

At large N_c the distribution amplitude is concentrated at

$$x_k \sim 1/N_c . \tag{5.7}$$

Therefore it is convenient to introduce new variables

$$y_k = N_c x_k \tag{5.8}$$

which behave as $O(N_c^0)$.

B. Generating functional

Now we define

$$\begin{aligned} \Phi(g) = N_c^{-N_c/2} &\int_0^{\infty} dy_1 \int_0^{\infty} dy_2 \ldots \int_0^{\infty} dy_{N_c} g_{f_1 s_1}(y_1) g_{f_2 s_2}(y_2) \ldots g_{f_{N_c} s_{N_c}}(y_{N_c}) \\ &\times \Psi_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})} \left(\frac{y_1}{N_c}, \frac{y_2}{N_c}, \ldots, \frac{y_{N_c}}{N_c} \right) . \end{aligned} \tag{5.9}$$

The factor $N_c^{-N_c/2}$ is inserted here in order to compensate the N_c growth of the contribution of the kinematical factor $(nP)^{N_c/2}$ coming to Eq. (5.5) from the product of N_c fields χ (5.4).

At large N_c we can write the general decomposition (1.4)

$$\Phi(g) = N_c^{\nu} A(g) \exp\left[N_c W(g)\right] \left[1 + O\left(N_c^{-1}\right)\right] . \tag{5.10}$$

From the definition (5.9) of $\Phi(g)$ it is obvious that for any constant λ we have

$$\Phi(\lambda g) = \lambda^{N_c} \Phi(g) . \tag{5.11}$$

Therefore

$$W(\lambda g) = W(g) + \ln \lambda , \tag{5.12}$$

$$A(\lambda g) = A(g) . \tag{5.13}$$

Differentiating identity (5.12) with respect to λ, we find

$$\sum_{fs} \int_0^{\infty} dy g_{fs}(y) \frac{\delta W(g)}{\delta g_{fs}(y)} = 1 . \tag{5.14}$$

The distribution amplitude contains the momentum conserving delta function:

$$\Psi_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}(x_1, x_2, \ldots, x_{N_c}) \sim \delta(x_1 + x_2 + \ldots + x_{N_c} - 1) \,. \tag{5.15}$$

Because to the presence of this delta function, the functional $\Phi(g)$ has the following property. For functions

$$g_{fs}^{(y_0,\beta)}(y) = \beta_{fs}\delta(y - y_0) \tag{5.16}$$

we have

$$\left|\Phi\left[g^{(y_0,\beta)}\right]\right| = \begin{cases} 0 & \text{if } y_0 \neq 1 \,, \\ \infty & \text{if } y_0 = 1 \,. \end{cases} \tag{5.17}$$

This corresponds to the following singularities of $W(g)$

$$\mathrm{Re}\, W\left[g^{(y_0,\beta)}\right] = \begin{cases} -\infty & \text{if } y_0 \neq 1 \,, \\ +\infty & \text{if } y_0 = 1 \,. \end{cases} \tag{5.18}$$

VI. EVOLUTION EQUATION

A. Evolution equation for the baryon distribution amplitude

The evolution equation for the baryon distribution amplitude at arbitrary N_c can be easily read from the literature starting from the original papers [5, 6]. However, one should be careful generalizing the standard equations written for the 3-quark distribution amplitude to the N_c-quark case. In large-N_c QCD the baryon distribution amplitude $\Psi(x_1, \ldots, x_{N_c})$ depends on on N_c variables x_k [one of them can be eliminated using the conservation of the momentum (5.15)]. In QCD with N_c colors, the dependence of the leading-twist baryon distribution amplitude Ψ^μ on the normalization point μ is described by the evolution equation

$$\mu\frac{\partial}{\partial\mu}\Psi^\mu(x_1, \ldots, x_{N_c}) = -\frac{N_c+1}{2N_c}\frac{\alpha_s(\mu)}{\pi}\sum_{1\leq i<j\leq N_c} K_{ij}\Psi^\mu(x_1, \ldots, x_{N_c}) \,. \tag{6.1}$$

We work in the leading order in the strong coupling $\alpha_s(\mu)$. We have omitted the flavor and spin indices concentrating on pair-interaction structure of the evolution kernel. The indices i, j of K_{ij} imply that this operator acts on the variables x_i, x_j. For example, the compact equation

$$\phi = K_{12}\psi \tag{6.2}$$

should be expanded as follows

$$\phi^{(f_1 s_1)(f_2 s_2)}(x_1, x_2) = \sum_{f_1' s_1' f_2' s_2'} \int dx_1' dx_2' K^{(f_1 s_1)(f_2 s_2)}_{(f_1' s_1')(f_2' s_2')}(x_1, x_2; x_1', x_2')\psi^{(f_1' s_1')(f_2' s_2')}(x_1', x_2') \,. \tag{6.3}$$

The kernel K is diagonal in both flavor f_k and helicity s_k:

$$K^{(f_1 s_1)(f_2 s_2)}_{(f_1' s_1')(f_2' s_2')}(x_1, x_2; x_1', x_2') = \delta^{f_1}_{f_1'}\delta^{f_2}_{f_2'}\delta^{s_1}_{s_1'}\delta^{s_2}_{s_2'}\tilde{K}^{s_1 s_2}(x_1, x_2; x_1', x_2') \,. \tag{6.4}$$

More information on the evolution kernel and its properties can be found in Appendix B.

B. Evolution equation for the generating functional

Now we want to consider the large-N_c limit. As was explained above, a consistent analysis of the large-N_c limit is possible in terms of the generating functional $\Phi(g)$ (5.9). Therefore the first step is to rewrite the evolution equation (6.1) in terms of the generating functional $\Phi(g)$. Let us contract Eq. (6.1) with the product of functions $g_{f_k s_k}(y_k)$

$$\begin{aligned} &\mu\frac{\partial}{\partial\mu}\left[\prod_{k=1}^{N_c}\int_0^\infty dy_k g_{f_k s_k}(y_k)\right]\Psi^\mu_{(f_1 s_1)\ldots(f_{N_c} s_{N_c})}\left(\frac{y_1}{N_c}, \ldots, \frac{y_{N_c}}{N_c}\right) \\ &= -\frac{N_c+1}{2N_c}\frac{\alpha_s(\mu)}{\pi}\sum_{1\leq i<j\leq N_c}\prod_{k=1}^{N_c}\left[\int_0^\infty dy_k g_{f_k s_k}(y_k)\right] \\ &\times \left[K_{ij}\Psi^\mu\right]_{(f_1 s_1)\ldots(f_{N_c} s_{N_c})}\left(\frac{y_1}{N_c}, \ldots, \frac{y_{N_c}}{N_c}\right) \,. \end{aligned} \tag{6.5}$$

Note that the operators K_{ij} are invariant under dilations. Therefore the transition from x_k to $y_k = N_c x_k$ does not change the form of these operators. All the $N_c(N_c-1)/2$ terms of the sum over $i<j$ on the RHS are identical and can be written in terms of the generating functional $\Phi(g)$

$$\begin{aligned}
&\sum_{1\leq i<j\leq N_c}\sum_{f_m s_m}\left[\prod_{k=1}^{N_c}\int_0^\infty dy_k g_{f_k s_k}(y_k)\right][K_{ij}\Psi^\mu]_{(f_1 s_1)\ldots(f_{N_c}s_{N_c})}\left(\frac{y_1}{N_c},\ldots,\frac{y_{N_c}}{N_c}\right)\\
&=\frac{N_c(N_c-1)}{2}\sum_{f_m s_m}\left[\prod_{k=1}^{N_c}\int_0^\infty dy_k g_{f_k s_k}(y_c)\right][K_{12}\Psi^\mu]_{(f_1 s_1)\ldots(f_{N_c}s_{N_c})}\left(\frac{y_1}{N_c},\ldots,\frac{y_{N_c}}{N_c}\right)\\
&=\frac{1}{2}\int_0^\infty dy_1\int_0^\infty dy_2\sum_{f_1 f_2 s_1 s_2} g_{f_1 s_1}(y_1)g_{f_2 s_2}(y_2)\left[K_{12}\frac{\delta}{\delta g_{f_1 s_1}(y_1)}\frac{\delta}{\delta g_{f_2 s_2}(y_2)}\Phi_\mu(g)\right].
\end{aligned}\tag{6.6}$$

Now the evolution equation (6.5) takes the form

$$\mu\frac{\partial}{\partial\mu}\Phi_\mu(g)=-\frac{N_c+1}{2N_c}\frac{\alpha_s(\mu)}{2\pi}\left\{(g\otimes g)\cdot K\cdot\left[\left(\frac{\delta}{\delta g}\otimes\frac{\delta}{\delta g}\right)\Phi_\mu(g)\right]\right\},\tag{6.7}$$

where we use the short notation

$$\begin{aligned}
&(g\otimes g)\cdot K\cdot\left[\left(\frac{\delta}{\delta g}\otimes\frac{\delta}{\delta g}\right)\Phi_\mu(g)\right]\\
&=\int_0^\infty dy_1\int_0^\infty dy_2\sum_{f_1 f_2 s_1 s_2} g_{f_1 s_1}(y_1)g_{f_2 s_2}(y_2)\left[K_{12}\frac{\delta}{\delta g_{f_1 s_1}(y_1)}\frac{\delta}{\delta g_{f_2 s_2}(y_2)}\Phi_\mu(g)\right]\\
&=\int_0^\infty dy_1\int_0^\infty dy_2\int_0^\infty dy_1'\int_0^\infty dy_2'\sum_{f_1 f_2 s_1 s_2}\\
&\times g_{f_1 s_1}(y_1)g_{f_2 s_2}(y_2)\tilde{K}^{s_1 s_2}(y_1,y_2;y_1',y_2')\frac{\delta}{\delta g_{f_1 s_1}(y_1')}\frac{\delta}{\delta g_{f_2 s_2}(y_2')}\Phi_\mu(g).
\end{aligned}\tag{6.8}$$

Here we used Eq. (6.4).

C. Evolution equation in the leading order of the large-N_c limit

Now we can study the large-N_c limit of the evolution equation (6.7). We insert the large-N_c ansatz (5.10) into Eq. (6.7) and obtain in the leading order of the $1/N_c$ expansion:

$$\mu\frac{\partial}{\partial\mu}W_\mu(g)=-a(\mu)\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W_\mu(g)}{\delta g}\otimes\frac{\delta W_\mu(g)}{\delta g}\right]\right\},\tag{6.9}$$

where

$$a(\mu)=\lim_{N_c\to\infty}\frac{\alpha_s(\mu)N_c}{4\pi}.\tag{6.10}$$

This limit exists since

$$\alpha_s(\mu)=O(N_c^{-1}).\tag{6.11}$$

Eq. (6.9) is obviously compatible with the constraint (5.12) on $W_\mu(g)$. The evolution equation (6.9) is nonlinear in $W_\mu(g)$. Being a first order differential equation in μ, it allows us (in principle) to find W_μ at any normalization point μ starting from the initial value for W_{μ_0} at some point μ_0.

Instead of μ it is convenient to introduce the new variable t such that

$$dt=2a(\mu)\frac{d\mu}{\mu}.\tag{6.12}$$

Then

$$\frac{\partial}{\partial t}W(g,t)=-\frac{1}{2}\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W(g,t)}{\delta g}\otimes\frac{\delta W(g,t)}{\delta g}\right]\right\}.\tag{6.13}$$

Note that the evolution equation (6.13) does not depend on the quantum numbers of the baryon. This agrees with the statement that functional $W_\mu(g)$ is universal for all low-lying baryons. The difference between baryons appears in the pre-exponential factor $A(g)$ (5.10).

D. Hamilton-Jacobi structure

Eq. (6.13) has the form of a functional Hamilton-Jacobi equation. Changing the notation

$$g \to q\,, \quad W(g,t) \to S(q,t)\,, \tag{6.14}$$

we easily see that we deal with the equation

$$\frac{\partial S}{\partial t} = -H\left(\frac{\partial S}{\partial q}, q\right)\,, \tag{6.15}$$

where the Hamiltonian (written in terms of discrete variables p_n, q_n)

$$H(p,q) = \frac{1}{2}\sum_{ijmn} K_{ijmn} q_i q_j p_m p_n \tag{6.16}$$

is quadratic both in coordinates q_n and in momenta

$$p_n = \frac{\partial S}{\partial q_n}\,. \tag{6.17}$$

E. Evolution equation in the next-to-leading order

Inserting ansatz (5.10) into the evolution equation (6.7), we can also derive an equation for the pre-exponential factor $A(g,t)$

$$\begin{aligned}\frac{\partial}{\partial t}\ln A(g,t) = &-\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W(g,t)}{\delta g}\otimes\frac{\delta \ln A(g,t)}{\delta g}\right]\right\}\\ &+b(t)\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W(g,t)}{\delta g}\otimes\frac{\delta W(g,t)}{\delta g}\right]\right\}\\ &-\frac{1}{2}\left\{(g\otimes g)\cdot K\cdot\left[\left(\frac{\delta}{\delta g}\otimes\frac{\delta}{\delta g}\right)W(g,t)\right]\right\}\,,\end{aligned} \tag{6.18}$$

where

$$b(t) = \frac{1}{2}\lim_{N_c\to\infty}\left\{N_c\left[1-\frac{N_c+1}{4\pi}\frac{\alpha_s(\mu)}{a(\mu)}\right]\right\}\,. \tag{6.19}$$

Taking the difference of two equations (6.18) for baryons B_1, B_2 described by the pre-exponential factors $A_{B_1}(g,t)$ and $A_{B_2}(g,t)$, we obtain the following equation for the evolution of $A_{B_1}(g,t)/A_{B_2}(g,t)$

$$\frac{\partial}{\partial t}\frac{A_{B_1}(g,t)}{A_{B_2}(g,t)} = -\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W(g,t)}{\delta g}\otimes\frac{\delta}{\delta g}\frac{A_{B_1}(g,t)}{A_{B_2}(g,t)}\right]\right\}\,. \tag{6.20}$$

This is a linear homogeneous equation for $A_{B_1}(g,t)/A_{B_2}(g,t)$. The structure of this equation agrees with the statement about the factorization of $A_B(g,t)$ into "elementary functionals" $\xi_i(g,t)$ [cf. Eq. (3.8)]:

$$A_B(g,t) = A_0(g,t)\prod_i\left[\xi_i(g,t)\right]^{n_i}\,. \tag{6.21}$$

Functionals $\xi_i(g,t)$ correspond to the elementary excitations ω_i in the mass formula (3.3). The factorization (6.21) is compatible with the evolution equation (6.20) if functionals $\xi_i(g,t)$ obey Eq. (6.20):

$$\frac{\partial}{\partial t}\xi_i(g,t) = -\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W(g,t)}{\delta g}\otimes\frac{\delta \xi_i(g,t)}{\delta g}\right]\right\}\,. \tag{6.22}$$

Strictly speaking, expression (6.21) must be modified by the zero mode factors which are discussed in Sec. X C.

VII. ASYMPTOTIC LIMIT

A. Double limit $N_c \to \infty$, $t \to \infty$

At finite N_c the asymptotic large-μ behavior of the baryon distribution amplitude is well known [see Eq. (8.30)]. In terms of the variable t (6.12) the limit of large scales μ corresponds to $t \to \infty$. Now we want to study the large-t behavior of the functional $W(g,t)$. Since the functional $W(g,t)$ is defined via the large-N_c limit, we face the problem of the double limit $N_c \to \infty$, $t \to \infty$. As we shall see, the order in which these two limits are taken is important. One of the results of our analysis of the double limit $N_c \to \infty$, $t \to \infty$ will be a general expression for the anomalous dimensions of the leading-twist baryon operators in the large N_c limit. Later (in Sec. IX) this expression will be used for the calculation of these anomalous dimensions.

B. Linearized theory of the asymptotic limit

Now we turn to the investigation of the asymptotic limit of $W(g,t)$ at $t \to \infty$. In this section the analysis will be based on the evolution equation (6.13). A complete systematic analysis of the nonlinear evolution equation (6.13) in the asymptotic regime is beyond the scope of this paper. Our aim is more pragmatic: we want to find the physical solution. On the way to this physical solution we shall use various assumptions about the structure of the solution. The justification of these intermediate assumptions comes from the final form of the solution and from the independent analysis of the asymptotic limit in Sec. VIII C.

We expect the following structure of the large-t behavior of the functional $W(g,t)$

$$W(g,t) \stackrel{t\to\infty}{\to} W_{\rm as}(g,t) \equiv W_0(g) - \sigma(t)\,, \tag{7.1}$$

where the function $\sigma(t)$ is g independent and the functional $W_0(g)$ does not depend on t.

Inserting the ansatz (7.1) into the evolution equation (6.13), we arrive at the system of equations

$$\frac{1}{2}\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta W_0(g)}{\delta g}\right]\right\} = E\,, \tag{7.2}$$

$$\sigma(t) = Et\,, \tag{7.3}$$

where E is some constant.

According to Eqs. (5.12) and (7.1) the functional $W_0(g)$ must obey the condition

$$W_0(\lambda g) = W_0(g) + \ln\lambda\,. \tag{7.4}$$

Note that in terms of the Hamilton-Jacobi interpretation (6.15) of the evolution equation (6.13) the asymptotic equations (7.2) and (7.3) correspond to fixing the energy E in the Hamilton-Jacobi equation:

$$H\left(\frac{\partial S}{\partial q}, q\right) = E\,, \quad \frac{\partial S}{\partial t} = -E\,. \tag{7.5}$$

This "energy" interpretation of the asymptotic regime is quite similar to the standard analysis of the asymptotic distribution amplitudes of hadrons in terms of the lowest eigenstates of the effective Hamiltonian describing the evolution. However, our current large-N_c evolution equation is nonlinear and this leads to certain complications.

The true physical functional $W(g,t)$ is different from the asymptotic solution $W_{\rm as}(g,t)$ (7.1):

$$W(g,t) = W_{\rm as}(g,t) + \Delta W(g,t)\,. \tag{7.6}$$

Both $W(g,t)$ and $W_{\rm as}(g,t)$ satisfy condition (5.12). Therefore we must have

$$\Delta W(\lambda g,t) = \Delta W(g,t)\,. \tag{7.7}$$

At large t the $\Delta W(g,t)$ becomes small and we can linearize the evolution equation (6.13) in $\Delta W(g,t)$:

$$\frac{\partial}{\partial t}\Delta W(g,t) = -\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W_{\rm as}(g,t)}{\delta g}\otimes\frac{\delta \Delta W(g,t)}{\delta g}\right]\right\}\,. \tag{7.8}$$

According to Eq. (7.1) we have

$$\frac{\delta W_{\rm as}(g,t)}{\delta g} = \frac{\delta W_0(g)}{\delta g}\,. \tag{7.9}$$

Therefore

$$\frac{\partial}{\partial t}\Delta W(g,t) = -\left\{(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta \Delta W(g,t)}{\delta g}\right]\right\}\,. \tag{7.10}$$

In order to solve this equation, let us first consider the spectral problem

$$(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta \xi_k(g)}{\delta g}\right] = \Omega_k\xi_k(g) \tag{7.11}$$

in the class of functionals $\xi_k(g)$ obeying condition (7.7)

$$\xi_k(\lambda g) = \xi_k(g)\,. \tag{7.12}$$

The solution of Eq. (7.10) can be represented in the form

$$\Delta W(g,t) = \sum_k a_k e^{-\Omega_k t}\xi_k(g)\,. \tag{7.13}$$

Now we insert Eqs. (7.1), (7.3), and (7.13) into Eq. (7.6):

$$W(g,t) \stackrel{t\to\infty}{=} W_0(g) - Et + \sum_k a_k e^{-\Omega_k t}\xi_k(g)\,. \tag{7.14}$$

This equation is derived in the double limit $N_c\to\infty$, $t\to\infty$. Note that the limit $N_c\to\infty$ is taken first [remember that the functional $W(g,t)$ is defined by Eq. (5.10) via the large-N_c limit].

C. Asymptotic behavior in terms of anomalous dimensions

Formally Eq. (7.14) solves the problem of the asymptotic limit $t\to\infty$. Nevertheless one can wonder how this solution is connected with the standard representation of the t dependence in terms of the decomposition in operators with given anomalous dimensions. For the generating functional $\Phi(g,t)$ (5.9) this decomposition has the form

$$\Phi(g,t) \stackrel{t\to\infty}{\to} \sum_\alpha \Phi_\alpha(g)\exp\left(-\Gamma_\alpha t\right)\,. \tag{7.15}$$

Here Γ_α are anomalous dimensions of operators diagonalizing the evolution. The expansion (7.15) is written for finite N_c. If we take the large-N_c limit in Eq. (7.15), then we must be careful about the order of the two limits $N_c\to\infty$ and $t\to\infty$. The approach based on Eq. (7.15) assumes that the limit of large t is taken before the limit $N_c\to\infty$. On the contrary, Eq. (7.14) deals with the opposite order of the two limits (first $N_c\to\infty$ and then $t\to\infty$).

At large N_c we expect the following structure of the anomalous dimensions Γ_α

$$\Gamma_\alpha = c_1 N_c + c_0(\alpha) + c_{-1}(\alpha)N_c^{-1} + O(N_c^{-2})\,. \tag{7.16}$$

Note that the leading order coefficient c_1 is independent of α. This independence will be explained later.

If we insert decomposition (7.16) into Eq. (7.15), then the $1/N_c$ expansion for Γ_α will be exponentiated:

$$\Phi(g,t) \stackrel{t\to\infty, N_c\to\infty}{\to} \sum_\alpha \Phi_\alpha(g)\exp\left\{-t\left[c_1N_c + c_0(\alpha) + c_{-1}(\alpha)N_c^{-1}+\ldots\right]\right\}\,. \tag{7.17}$$

This exponentiation of the $1/N_c$ expansion for Γ_α means that we have a rather nontrivial sensitivity to the order of the limits $t\to\infty$ and $N_c\to\infty$.

If one is interested in the other order of limits (first $N_c \to \infty$ and then $t \to \infty$), then one should start from Eq. (7.14). Inserting the decomposition (7.14) into Eq. (5.10), we find

$$\Phi(g,t) \stackrel{N_c\to\infty,\, t\to\infty}{\to} A(g,t) \exp\left\{ N_c \left[W_0(g) - Et + \sum_k a_k e^{-\Omega_k t} \xi_k(g) \right] \right\} . \tag{7.18}$$

Now we can compare the two asymptotic expressions (7.17), (7.18) derived for functional $\Phi(g,t)$ in the double limit $N_c \to \infty$, $t \to \infty$ assuming different orders of these two limits. Separating the factors containing the exponentiated product $N_c t$, we conclude from the comparison of Eqs. (7.17), (7.18) that

$$e^{-c_1 t N_c} = e^{-E t N_c} . \tag{7.19}$$

We see that

$$c_1 = E . \tag{7.20}$$

This result explains why the coefficient c_1 appearing in large-N_c expansion (7.16) for Γ_α is independent of α.

One can also match the two asymptotic representations (7.17) and (7.18) in the next order of the $1/N_c$ expansion. To this aim we expand the RHS of Eq. (7.18) in powers of $a_k e^{-\Omega_k t} \xi_k(g)$:

$$\begin{aligned} \Phi(g,t) &\stackrel{N_c\to\infty,\, t\to\infty}{\to} A(g,t) \exp\{N_c W_0(g)\} \\ &\times \sum_{\{n_k\}} \exp\left[-\left(N_c E + \sum_k n_k \Omega_k \right) t \right] \prod_k \frac{1}{n_k!} \left[N_c a_k \xi_k(g) \right]^{n_k} . \end{aligned} \tag{7.21}$$

Comparing this decomposition with Eq. (7.17), we see that

$$c_0(\alpha) = \Delta E + \sum_k n_k \Omega_k \quad (n_k = 0, 1, 2, \ldots) , \tag{7.22}$$

where the $O(N_c^0)$ contribution ΔE comes from exponential part of the large-t asymptotic behavior of $A(g,t) \sim e^{-t\Delta E}$. Inserting this expression for $c_0(\alpha)$ into Eq. (7.16) and using the expression (7.20) for c_1, we find

$$\Gamma_{\{n_k\}} = N_c E + \left(\Delta E + \sum_k n_k \Omega_k \right) + c_{-1}(n_1, n_2, \ldots) N_c^{-1} + O(N_c^{-2}) . \tag{7.23}$$

Here we label the anomalous dimensions Γ_α with sets of integer numbers $\{n_k\}$:

$$\alpha = \{n_k\} \quad (n_k = 0, 1, 2, \ldots) . \tag{7.24}$$

At finite N_c the spectrum of the anomalous dimensions comes from the diagonalization of the N_c-particle "effective Hamiltonian" and the anomalous dimensions Γ_α are labeled by $N_c - 1$ "quantum numbers" $\alpha_1, \alpha_2, \ldots \alpha_{N_c-1}$ (one degree of freedom is eliminated by the momentum conservation). In the limit $N_c \to \infty$ we arrive at the "oscillator spectrum" (7.23) and the anomalous dimensions are parametrized by a set of integer excitation numbers n_k.

The dependence of the anomalous dimensions on some quantum numbers (e.g. helicity) appears only in the order $O(N_c^{-1})$. Denoting these quantum numbers by ρ, we can write the correct generalization of Eq. (7.23)

$$\Gamma_{\{n_k,\rho\}} = N_c E + \left(\Delta E + \sum_k n_k \Omega_k \right) + c_{-1}(n_1, n_2, \ldots; \rho) N_c^{-1} + O(N_c^{-2}) . \tag{7.25}$$

This structure of the spectrum corresponding to an effective harmonic oscillator in the order N_c^0 (with anharmonic corrections appearing in the next orders of the $1/N_c$ expansion) is typical for semiclassical systems. The semiclassical nature of the $1/N_c$ expansion is known since long ago. Depending on the methods used for the construction of the $1/N_c$ expansion, the semiclassical features can arise via the mean field approximation, saddle point approximation in path integrals, WKB expansion etc.

Our derivation of the expansion (7.25) was based on the asymptotic linearized analysis of the Hamilton-Jacobi equation (7.10) in terms of the large-t decomposition (7.14). In this approach the spectrum of Ω_k is given by the eigenvalues of the spectral problem (7.11) for the functionals $\xi_k(g)$.

In the rest of the paper we compute parameters $E, \Delta E$ and Ω_k appearing in the expansion (7.25). The results are given by Eq. (8.9) for E and Eq. (9.4) for ΔE. The parameters Ω_k are computed in Eqs. (9.40), (9.41) in the case of one quark flavor $N_f = 1$. The role of parameter ρ is played by the helicity J_3 in this case. The $O(N_c^{-1})$ term $c_{-1}(0, 0, \ldots; J_3) N_c^{-1}$ is computed for the lowest anomalous dimension corresponding to the asymptotic wave function of baryons with given helicity J_3 [see Eq. (9.2)].

VIII. CALCULATION OF THE ASYMPTOTIC FUNCTIONAL $W_0(g)$

A. Methods

We want to compute the asymptotic functional $W_0(g)$ which determines the asymptotic behavior (7.1) of $W(g,t)$ at large t. Two methods can be used for the calculation:

1) We can solve Eq. (7.2).

2) We can use the traditional approach to the determination of the large-t asymptotic baryon distribution amplitude at finite N_c. Using this finite-N_c asymptotic distribution amplitude we can construct the corresponding generating functional by analogy with Eq. (5.9). The large-N_c asymptotic behavior of this functional will be dominated by the exponential term containing information on $W_0(g)$.

The first method is described in Sec. VIII B. In Sec. VIII C we show how the second method works. In Sec. VIII D we compare these two methods. The computed functional $W_0(g)$ depends on the number of quark flavors N_f. The nature of this dependence is studied in Sec. VIII E. In Sec. VIII F we comment on the analytical properties of the functional $W(g,t)$ and illustrate the general statements using our result for the asymptotic functional $W_0(g)$.

B. Solution of the equation for $W_0(g)$

According to Eq. (7.1) the main quantity characterizing the asymptotic limit is the functional $W_0(g)$. This functional can be found by solving Eq. (7.2).

Our approach to Eq. (7.2) will be rather heuristic. We simply use an ansatz which allows us to solve Eq. (7.2) for some special value of E. The relevance of this value of E for the description of the asymptotic behavior will be seen from the analysis of Sec. VIII C. The form of the solution $W_0(g)$ of Eq. (7.2) depends on the number of quark flavors N_f. We start from the simplest case $N_f = 1$. The result found for $N_f = 1$ is easily generalized to the more interesting case $N_f = 2$.

1. Case $N_f = 1$

In order to solve Eq. (7.2) we use the property of the evolution kernel $\tilde{K}^{s_1 s_2}$

$$\int dy_1' \int dy_2' \tilde{K}^{s_1 s_2}(y_1, y_2; y_1', y_2') y_1' y_2' e^{-X(g)(y_1'+y_2')} = \frac{1}{2}\delta^{s_1 s_2} y_1 y_2 e^{-X(g)(y_1+y_2)} \tag{8.1}$$

which is valid for any functional $X(g)$. This identity follows from Eq. (B21) derived in Appendix B. Keeping in mind this identity, we make the following ansatz

$$\frac{\delta W_0(g)}{\delta g_s(y)} = y e^{-X(g)y} U_s(g)\,, \tag{8.2}$$

where $X(g)$ and $U_s(g)$ are some functionals of g_s. Inserting this ansatz into Eq. (7.2) and using Eq. (8.1), we find

$$\sum_s \left[G_s(g) U_s(g)\right]^2 = 4E\,, \tag{8.3}$$

where

$$G_s(g) \equiv \int_0^\infty g_s(y) y e^{-X(g)y}\,. \tag{8.4}$$

Combining this expression for $G_s(g)$ with Eq. (8.2), we find

$$\int_0^\infty dy g_s(y) \frac{\delta W_0(g)}{\delta g_s(y)} = G_s(g) U_s(g)\,. \tag{8.5}$$

According to Eq. (5.14) we have

$$\sum_s \int_0^\infty dy g_s(y) \frac{\delta W_0(g)}{\delta g_s(y)} = 1\,. \tag{8.6}$$

Combining the last two equations, we conclude that

$$\sum_s G_s(g)U_s(g) = 1\,. \tag{8.7}$$

We can satisfy both Eq. (8.3) and Eq. (8.7) by taking

$$G_s(g)U_s(g) = \frac{1}{2}\,, \tag{8.8}$$

which corresponds to

$$E = \frac{1}{8}\,. \tag{8.9}$$

Now it follows from Eqs. (8.2), (8.4), and (8.8)

$$\frac{\delta W_0(g)}{\delta g_s(y)} = \frac{1}{2} y e^{-X(g)y} \left[\int_0^\infty dy' g_s(y') y' e^{-X(g)y'}\right]^{-1}\,. \tag{8.10}$$

We can look for the solution of this equation in the form

$$W_0(g) = F\left[X(g)\right] + \frac{1}{2}\sum_s \ln\left[\int_0^\infty dy g_s(y) y e^{-X(g)y}\right]\,, \tag{8.11}$$

where $F(X)$ is an arbitrary function and the functional $X(g)$ is implicitly determined by the equation

$$\frac{\partial}{\partial X}\left\{F(X) + \frac{1}{2}\sum_s \ln\left[\int_0^\infty dy g_s(y) y e^{-Xy}\right]\right\} = 0 \quad\Longrightarrow\quad X = X(g)\,. \tag{8.12}$$

Thus we have found an infinite set of solutions $W_0(g)$ of Eq. (7.2) corresponding to the same constant E (8.9). These solutions are parametrized by arbitrary functions $F(X)$. The physical solution is fixed by the condition (5.18) which leads to the following choice of $F(X)$:

$$F(X) = X\,. \tag{8.13}$$

Indeed, taking this function $F(X)$ and computing the LHS of Eq. (8.12) for functions $g^{(y_0,\beta)}$ (5.16), we obtain

$$\frac{\partial}{\partial X}\left\{F(X) + \frac{1}{2}\sum_s \ln\left[\int_0^\infty dy g_s^{(y_0,\beta)}(y) y e^{-Xy}\right]\right\} = 1 - y_0\,. \tag{8.14}$$

Therefore Eq. (8.12) has no solutions at $y_0 \neq 1$ in agreement with the condition (5.18).

We can combine Eqs. (8.11), (8.12), and (8.13) into the final representation for $W_0(g)$

$$W_0(g) = \underset{X}{\mathrm{extremum}}\left\{X + \frac{1}{2}\sum_s \ln\left[\int_0^\infty dy g_s(y) y e^{-Xy}\right]\right\}\,. \tag{8.15}$$

The word *extremum* must be understood in the sense of the equation

$$\frac{\partial}{\partial X}\left\{X + \frac{1}{2}\sum_s \ln\left[\int_0^\infty dy g_s(y) y e^{-Xy}\right]\right\} = 0\,. \tag{8.16}$$

In the general case both g_s and X can be complex so that we cannot speak about a maximum or a minimum. Equation (8.16) determines the functional $X(g)$ and this functional should be used on the RHS of Eq. (8.15) for the calculation of $W_0(g)$.

2. Case $N_f = 2$

The generalization of the above results to the case $N_f = 2$ is straightforward. By analogy with Eq. (8.2) we make the ansatz

$$\frac{\delta W_0(g)}{\delta g_{fs}(y)} = y e^{-X(g)y} U_{fs}(g)\,. \tag{8.17}$$

Introducing the notation

$$G_{fs}(g) \equiv \int_0^\infty dy g_{fs}(y) y e^{-X(g)y}\,, \tag{8.18}$$

we find the $N_f = 2$ analogs of Eqs. (8.3) and (8.8)

$$\sum_{f_1 f_2 s} G_{f_1 s}(g) G_{f_2 s}(g) U_{f_1 s}(g) U_{f_2 s}(g) = 4E\,, \tag{8.19}$$

$$\sum_{fs} G_{fs}(g) U_{fs}(g) = 1\,. \tag{8.20}$$

One can satisfy both equations by taking

$$\sum_f G_{fs_1}(g) U_{fs_2}(g) = \frac{1}{2}\delta_{s_1 s_2}\,, \tag{8.21}$$

which leads to the same value

$$E = \frac{1}{8} \tag{8.22}$$

as in the $N_f = 1$ case (8.9).

According to Eqs. (8.18) and (8.21) we have

$$U_{fs}(g) = \frac{1}{2}\left[G(g)\right]^{-1}_{sf} = \frac{1}{2}\left\{\left[\int_0^\infty dy g(y) y e^{-X(g)y}\right]^{-1}\right\}_{sf}\,. \tag{8.23}$$

Inserting this result into Eq. (8.17), we find

$$\frac{\delta W_0(g)}{\delta g_{fs}(y)} = \frac{1}{2} y e^{-X(g)y}\left\{\left[\int_0^\infty dy' g(y') y e^{-X(g)y'}\right]^{-1}\right\}_{sf}\,. \tag{8.24}$$

The general solution of this equation compatible with the momentum conservation constraint (5.18) is

$$W_0(g) = \operatorname*{extremum}_X \left\{X + \frac{1}{2}\ln\det_{fs}\left[\int_0^\infty dy g_{fs}(y) y e^{-Xy}\right]\right\}\,. \tag{8.25}$$

Similarly to the $N_f = 1$ case, this compact representation should be understood in the sense of the (generally speaking complex) "extremum equation"

$$\frac{\partial}{\partial X}\left\{X + \frac{1}{2}\ln\det_{fs}\left[\int_0^\infty dy g_{fs}(y) y e^{-Xy}\right]\right\} = 0\,, \tag{8.26}$$

which determines the functional $X(g)$. Eq. (8.26) can be rewritten in the form

$$\mathrm{Tr}\left\{\left[\int_0^\infty dy g(y) y^2 e^{-Xy}\right]\left[\int_0^\infty dy' g(y') y' e^{-Xy'}\right]^{-1}\right\} = 2\,. \tag{8.27}$$

Inserting Eq. (8.25) into Eq. (7.1), we find the final expression for the asymptotic functional:

$$W_{\mathrm{as}}(g,t) = \operatorname*{extremum}_X \left\{X + \frac{1}{2}\ln\det_{fs}\left[\int_0^\infty dy g_{fs}(y) y e^{-Xy}\right]\right\} - Et\,. \tag{8.28}$$

C. Direct construction of the asymptotic functional

Let us consider the case of $N_f = 2$ flavors. The asymptotic distribution amplitude of baryons with helicity

$$|J_3| \leq T \tag{8.29}$$

is

$$\begin{aligned} &\Psi^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}(x_1, x_2, \ldots, x_{N_c}) \\ &= C^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})} x_1 x_2 \ldots x_{N_c} \delta(x_1 + x_2 + \ldots + x_{N_c} - 1)\,. \end{aligned} \tag{8.30}$$

The helicity-flavor part $C^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}$ coincides with the wave function of the toy quark model (4.2) up to a normalization factor:

$$C^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})} = \tilde{a}^{(B)} N_c^{3N_c/2} \psi^{T T_3 J_3}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}\,. \tag{8.31}$$

The factor of $N_c^{3N_c/2}$ is inserted in order to simplify the subsequent expressions (the question about the large-N_c behavior of the coefficient $\tilde{a}^{(B)}$ requires a separate analysis).

Combining Eqs. (4.2), (8.30), and (8.31), we find

$$\begin{aligned} &\Psi^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}(x_1, x_2, \ldots, x_{N_c}) = \tilde{a}^{(B)} c_{T N_c} N_c^{3N_c/2} \\ &\times \left[\int dR D^{T}_{J_3 T_3}(R^{-1}) \prod_{k=1}^{N_c} R_{f_k s_k} \right] x_1 x_2 \ldots x_{N_c} \delta(x_1 + x_2 + \ldots + x_{N_c} - 1)\,. \end{aligned} \tag{8.32}$$

It is easy to see that this wave function diagonalizes the RHS of the evolution equation (6.1). Indeed, using identity (B20), we find

$$K_{ij} \Psi^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})} = \frac{1}{2} \delta_{s_i s_j} \Psi^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}\,. \tag{8.33}$$

The helicity-flavor wave function $C^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}$ of the baryon with helicity J_3 has $\frac{N_c}{2} + J_3$ indices $s_k = +1/2$ and $\frac{N_c}{2} - J_3$ indices $s_k = -1/2$. Therefore

$$\begin{aligned} &\sum_{1 \leq i < j \leq N_c} K_{ij} \Psi^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})} = \sum_{1 \leq i < j \leq N_c} \frac{1}{2} \delta_{s_i s_j} \Psi^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})} \\ &= \frac{1}{2} \left[\frac{\left(\frac{N_c}{2} + J_3\right)\left(\frac{N_c}{2} + J_3 - 1\right)}{2} + \frac{\left(\frac{N_c}{2} - J_3\right)\left(\frac{N_c}{2} - J_3 - 1\right)}{2} \right] \Psi^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})} \\ &= \frac{1}{2} \left[\frac{N_c}{2} \left(\frac{N_c}{2} - 1 \right) + (J_3)^2 \right] \Psi^{(B)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}\,. \end{aligned} \tag{8.34}$$

Therefore the evolution equation (6.1) for the wave function (8.30) becomes

$$\mu \frac{\partial}{\partial \mu} \Psi^{\mu}(x_1, \ldots, x_{N_c}) = -\frac{N_c + 1}{2N_c} \frac{\alpha_s(\mu)}{2\pi} \left[\frac{N_c}{2} \left(\frac{N_c}{2} - 1 \right) + (J_3)^2 \right] \Psi^{\mu}(x_1, \ldots, x_{N_c}). \tag{8.35}$$

Taking the limit of large N_c and assuming that $J_3 = O(N_c^0)$ in this limit, we find

$$\mu \frac{\partial}{\partial \mu} \Psi^{\mu}(x_1, \ldots, x_{N_c}) = -\frac{N_c^2}{16\pi} \alpha_s(\mu) \Psi^{\mu}(x_1, \ldots, x_{N_c})\,. \tag{8.36}$$

This corresponds to the μ dependence

$$\Psi^{\mu}(x_1, \ldots, x_{N_c}) = \Psi^{\mu_0}(x_1, \ldots, x_{N_c}) \exp\left\{ N_c \left[\sigma(\mu_0) - \sigma(\mu) \right] \right\}\,, \tag{8.37}$$

where function $\sigma(\mu)$ obeys the equation

$$\mu \frac{\partial \sigma(\mu)}{\partial \mu} = \frac{N_c}{16\pi} \alpha_s(\mu)\,. \tag{8.38}$$

Changing from μ to the variable t (6.12) we find

$$\sigma = \frac{1}{8} t \,. \tag{8.39}$$

This agrees with the above results (7.3) and (8.22).

Thus the evolution of the asymptotic distribution amplitude (8.32) is described by the μ dependent factor

$$\tilde{a}^{(B)}(\mu) = a^{(B)} \exp\left[-N_c \sigma(\mu)\right] \,, \tag{8.40}$$

so that

$$\begin{aligned}
&\Psi^{(B,\mu)}_{(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}(x_1, x_2, \ldots, x_{N_c}) = a^{(B)} c_{TN_c} N_c^{3N_c/2} \exp\left[-N_c \sigma(\mu)\right] \\
&\times \left[\int dR D^T_{J_3 T_3}(R^{-1}) \prod_{k=1}^{N_c} R_{f_k s_k} \right] x_1 x_2 \ldots x_{N_c} \delta(x_1 + x_2 + \ldots + x_{N_c} - 1) \\
&= a^{(B)} c_{TN_c} N_c^{(3N_c/2)+1} \exp\left[-N_c \sigma(\mu)\right] \int dR D^T_{J_3 T_3}(R^{-1}) \int \frac{dZ}{2\pi} e^{-iZN_c} \prod_{k=1}^{N_c} \left(R_{f_k s_k} x_k e^{iZ x_k N_c} \right) \,.
\end{aligned} \tag{8.41}$$

Next we compute the generating functional (5.9) for this wave function

$$\begin{aligned}
\Phi_\mu(g) &\stackrel{\mu\to\infty}{=} \frac{a^{(B)} c_{TN_c} N_c}{2\pi} \exp\left[-N_c \sigma(\mu)\right] \int dR \int dZ D^T_{J_3 T_3}(R^{-1}) \\
&\times \exp\left\{ N_c \left\{ -iZ + \ln\left[R_{fs} \int_0^\infty dy g_{fs}(y) y e^{iZy} \right] \right\} \right\} \,.
\end{aligned} \tag{8.42}$$

The integrals over R and Z can be taken using the saddle point method. Let us first perform the integration over R at fixed Z. This can be done by repeating the calculation of Sec. IV C for the quark model. We must simply replace

$$g \to \int_0^\infty dy g(y) y e^{iZy} \tag{8.43}$$

in Eqs. (4.6) and (4.21). As a result, we find the saddle point for the R integral in Eq. (8.42):

$$R^{-1} = \pm \left[\int_0^\infty dy g(y) y e^{iZy} \right]^{\mathrm{tr}} \left\{ \det\left[\int_0^\infty dy g(y) y e^{iZy} \right] \right\}^{-1/2} \,. \tag{8.44}$$

Inserting this result into the exponent of (8.42), we find with the exponential accuracy (neglecting factors independent of g and μ)

$$\begin{aligned}
\Phi^{\mathrm{as}}_\mu(g) &\sim \exp\left[-N_c \sigma(\mu)\right] \\
&\times \int dZ \exp\left\{ N_c \left\{ -iZ + \frac{1}{2} \ln\det\left[\int_0^\infty dy g(y) y e^{iZy} \right] \right\} \right\} \,.
\end{aligned} \tag{8.45}$$

The saddle point equation for the Z integral is

$$\frac{\partial}{\partial Z} \left\{ -iZ + \frac{1}{2} \ln\det\left[\int_0^\infty dy g(y) y e^{iZy} \right] \right\} = 0 \,. \tag{8.46}$$

This equation coincides with Eq. (8.26) if we set

$$X = -iZ \,. \tag{8.47}$$

Now we see that the saddle point integration in Eq. (8.45) reproduces our old results (8.25) and (8.28).

D. Comparison of the two methods

Thus the two methods described in Sections VIII B and VIII C have led us to the same result (8.25) for the functional $W_0(g)$. Note that in the second method we assumed that the baryon obeys the condition $|J_3| \leq T$ which was needed in order to make use of the nonrelativistic spin-flavor wave functions (8.31). In principle the condition $|J_3| \leq T$ is automatically satisfied for the lowest $O(N_c^{-1})$ excited baryons (3.2) with $J = T$. But the discussion of the universality of the functional $W(g)$ in Sec. III shows that the same functional $W(g)$ must describe not only the lowest $J = T$ baryons (3.2) but also higher $O(N_c^0)$ excitations (3.3) and baryon-meson scattering states. Note that the first method based on the asymptotic analysis of the evolution equation for $W(g)$ is completely compatible with the universality of $W(g)$. Actually the condition $|J_3| \leq T$ used in the second method can be omitted. Indeed, the origin of this condition is the totally symmetric form of the x_k dependent part of the asymptotic wave function (8.30). In the case of the baryons with $|J_3| > T$ we cannot keep this form of the x_k dependence. But one can construct the corresponding asymptotic wave function and show that it is "almost symmetric" in the sense that in the large-N_c limit this wave function leads to the generating functional $\Phi(g)$ which has the same $W(g)$ exponent, and all effects of the asymmetry of the wave function are localized in the pre-exponential factor $A(g)$.

One of the advantages of the second method considered in Sec. VIII C is that it allows us to trace the connection with the traditional analysis of the asymptotic baryon distribution amplitude. Another good feature of this method is that it clarifies the saddle point origin (8.44), (8.46) of Eqs. (8.24), (8.26) which appear rather formally in the first method. Concerning this saddle point interpretation, we must make an important comment. In principle the large-N_c limit justifies the applicability of the saddle point method to the calculation of the integral (8.42). However, the large-N_c limit *does not guarantee* that

1) the saddle point equation (8.46) has one and only one solution,

2) the R and Z "integration contours" can be properly deformed so that the solution of the saddle point equation gives the dominant contribution.

Generally speaking, the validity of the conditions needed for the applicability of the saddle point approximation depends on the function g for which we compute the asymptotic functional $W_0(g)$. It is easy to construct examples of functions g for which the saddle point method works and counter-examples illustrating the violation of the saddle point method.

To summarize, our representation for the asymptotic functional (8.28) should be considered as a formal expression that carries information about the saddle point equation but does not fix the relevant solution of this equation.

E. N_f dependence

The results of our analysis of the cases $N_f = 1$ and $N_f = 2$ are represented by expressions (8.15) and (8.25) for the functional $W_0(g)$. We stress that the $N_f = 1$ and $N_f = 2$ results are different. Let us use the notation $W_0^{(N_f)}(g^{(N_f)})$ in order to mark the N_f dependence. Starting from some $g_s^{(1)}$, let us define

$$g_{fs}^{(2)} = \begin{cases} g_s^{(1)} & \text{if } f = 1, \\ 0 & \text{if } f = 2 \,. \end{cases} \tag{8.48}$$

One could naively expect that this choice of $g_{fs}^{(2)}$ selects only the contribution of the quarks with $f = 1$ so that the values $W_0^{(2)}(g^{(2)})$ and $W_0^{(1)}(g^{(1)})$ must coincide. But this does not happen. Moreover, the functional $W_0^{(2)}(g^{(2)})$ (8.25) has a singularity for the functions (8.48) because of the vanishing determinant

$$\det_{fs} \left[\int_0^\infty dy g_{fs}^{(2)}(y) y e^{-Xy} \right] = 0 \,, \tag{8.49}$$

$$\mathrm{Re}\, W_0^{(2)}(g^{(2)}) = -\infty \,. \tag{8.50}$$

What stands behind this difference of the $N_f = 1$ and $N_f = 2$ cases? Obviously the N_f dependence of the nonperturbative dynamics of QCD has nothing to do with this problem because the evolution equation does not know anything about this nonperturbative dynamics. This difference has another origin.

The large-N_c baryons which we studied in the $N_f = 1$ case consist of quarks with the same flavor u. If we "embed" this $(uu \ldots u)$ baryon into the $N_f = 2$ theory, then it will be classified as a state with isospin $T_3 = N_c/2$. Thus our

large-N_c work in the $N_f = 1$ sector should be interpreted in the $N_f = 2$ terms as the limit

$$N_c \to \infty\,, \quad T_3 = \frac{N_c}{2} \to \infty\,. \tag{8.51}$$

However, the functional $W_0^{(2)}(g^{(2)})$ was introduced for another limit:

$$N_c \to \infty\,, \quad T = O(N_c^0)\,. \tag{8.52}$$

Although we did not emphasize the condition $T = O(N_c^0)$, it was really essential in our $N_f = 2$ analysis. For example, in the saddle point calculation of the R integral (8.42) we did not include the function $D^T_{J_3T_3}(R^{-1})$ into the saddle point equation, which is allowed only if T is kept fixed in the limit $N_c \to \infty$.

Now we understand the meaning of the singularity (8.50). After the exponentiation this singularity leads to zero:

$$\Phi^{(2)}(g^{(2)}) = A^{(2)}(g^{(2)}) \exp\left[N_c\, W_0^{(2)}(g^{(2)})\right] = 0\,. \tag{8.53}$$

The origin of this zero is obvious: function $g^{(2)}_{fs}$ (8.48) selects the states with $T_3 = N_c/2$ whereas the baryon has a finite isospin which is kept fixed at large N_c. This discrepancy leads to the vanishing functional $\Phi^{(2)}(g^{(2)})$.

Thus our analysis of the two cases $N_f = 1$ and $N_f = 2$ used different assumptions about the behavior of the baryon isospin at large N_c.

Now we can turn to the case of $N_f = 3$ flavors. Our experience with the isospin shows that one has to distinguish between two cases:

1) fixed strangeness $S = O(N_c^0)$,

2) growing strangeness $S = O(N_c)$.

The case of fixed strangeness can be trivially reduced to the $N_f = 2$ case. This follows from the universality of the generating functional $W(g)$. Indeed, in the large-N_c world with $N_f = 3$, the nonstrange baryons and baryons with $O(N_c^0)$ strangeness are described by the same functional $W^{(3)}(g^{(3)})$. For the nonstrange baryons the functional $\Phi^{(3)}(g^{(3)})$ (5.9) obviously does not depend on the strange components of $g^{(3)}_{fs}$ with $f = 3$. Now we apply Eq. (5.10) to these $g^{(3)}_{3s}$-independent functionals $\Phi^{(3)}(g^{(3)})$ and conclude that the functional $W^{(3)}(g^{(3)})$ is also $g^{(3)}_{3s}$-independent. Due to the universality this property of $W^{(3)}(g^{(3)})$ can be extended to the case of baryons with $O(N_c^0)$ strangeness. Thus in the $N_f = 3$ world, the baryons with $O(N_c^0)$ strangeness are described by the functional $W^{(3)}(g^{(3)})$ which is independent of the components $g^{(3)}_{3s}$. The dependence on $g^{(3)}_{3s}$ appears only in the pre-exponential functionals $A^{(3)}_B(g^{(3)})$.

F. Analytical properties of $W(g)$

According to Eq. (5.9) $\Phi(g)$ is a homogeneous polynomial functional of g. In particular, $\Phi(g)$ is a holomorphic functional of g. Does this mean that the functional $W(g)$ defined by Eq. (5.10) is also holomorphic in g? The situation is rather subtle. The expression for $W(g)$ via $\Phi(g)$ following from Eq. (5.10),

$$W(g) = \lim_{N_c\to\infty} \frac{1}{N_c} \ln \Phi(g)\,, \tag{8.54}$$

contains a logarithm which can lead to a violation of the analyticity. In principle, the appearance of this logarithm is a mere notational effect: instead of Eq. (5.10) we could work with the large-N_c asymptotic representation

$$\Phi_B(g, N_c) = N_c^\nu A_B(g)\, [E(g)]^{N_c} \left[1 + O(N_c^{-1})\right]\,. \tag{8.55}$$

Obviously

$$E(g) \equiv e^{W(g)}\,. \tag{8.56}$$

Using Eq. (8.55), we find

$$[E(g)]^2 = \lim_{N_c\to\infty} \frac{\Phi_B(g, N_c + 2)}{\Phi_B(g, N_c)}\,. \tag{8.57}$$

Here N_c is shifted by two in the numerator of the RHS. This is necessary for the cancellation of the factors $A_B(g)$. Indeed, according to Eq. (3.1) the quantum numbers of baryons B are different in theories with odd and even N_c. Eq. (8.57) shows that the functional $[E(g)]^2$ is well defined and has no ambiguities in contrast to the functional $W(g)$.

In order to illustrate these problems let us consider the toy quark model described in Sec. IV. Our result (4.10) for the toy analog $w(g)$ of the functional $W(g)$ contains a logarithmic singularity for degenerate matrices g with $\det g = 0$. According to Eq. (4.10) in the toy model we have the analog $\exp w(g)$ of the functional (8.56)

$$e(g) \equiv \exp w(g) = \sqrt{2 \det g}\,. \tag{8.58}$$

We see that the analyticity of $e(g)$ is broken by the square root singularity that appears for degenerate matrices g. As was mentioned in Sec. IV, the sign uncertainties of this square root are insignificant for $[e(g)]^{N_c}$ at even N_c. For odd N_c this sign ambiguity is compensated by a similar root ambiguity of the pre-exponential D function in Eq. (4.12).

Now let us turn to the analytical properties of the functional $W_0(g)$ describing the asymptotic behavior of the baryon distribution amplitude at high normalization points $\mu \to \infty$. To be specific, let us consider the case $N_f = 2$ when $W_0(g)$ is given by Eq. (8.25). It is easy to see that this expression for $W_0(g)$ has two ambiguities.

1) Expression (8.25) contains the logarithmic term

$$\frac{1}{2} \ln \det_{fs} \left[\int_0^\infty dy g_{fs}(y) y e^{-Xy} \right] \tag{8.59}$$

which has an additive $\pi i n$ ambiguity. This logarithm is accompanied by the coefficient $1/2$ so that the uncertainties of $\exp\left[N_c W_0(g)\right]$ reduce to $(\pm 1)^{N_c}$ and disappear in $\Phi_B(g)$ in the same way like in the case of the toy quark model.

2) Parameter X in Eq. (8.25) is defined implicitly via the extremum equation (8.26). The ambiguities of the determination of X from this extremum equation were discussed in Sec. VIII D. These ambiguities can also lead to a violation of the analyticity of the functional $W_0(g)$.

Note that the appearance of singularities in $W_0(g)$ and the violation of analyticity are natural from the point of view of the WKB method used in our large-N_c analysis. Such well-known "singular" features of the WKB method like caustics, Stokes lines, etc. have their analogs in our problem of the baryon wave function at large N_c.

IX. DIAGONALIZATION OF ANOMALOUS DIMENSIONS

A. Anomalous dimensions of baryon operators at large N_c

We already know that at large N_c the anomalous dimensions $\Gamma_{\{n_k\}}$ of baryonic operators are given by the relation (7.25). Parameter E appearing in this equation is determined by Eq. (8.9) so that

$$\Gamma_{\{n_k, J_3\}} = \frac{1}{8} N_c + \left(\Delta E + \sum_k n_k \Omega_k \right) + c_{-1}(n_1, n_2, \ldots; J_3) N_c^{-1} + O(N_c^{-2})\,. \tag{9.1}$$

Here J_3 is the helicity of the corresponding operator. In the case of one quark flavor $N_f = 1$, the helicity J_3 is the only quantum number which does not contribute in the order $O(N_c^0)$.

The lowest anomalous dimension corresponds to the case when all numbers n_k vanish. We know the corresponding anomalous dimension exactly from the evolution equation for the exact asymptotic wave function (8.35):

$$\Gamma_{\{n_k=0\}} = \frac{1}{4}\left(\frac{N_c}{2} - 1\right) + \frac{(J_3)^2}{2N_c}\,. \tag{9.2}$$

Note that we do not include the factor

$$\frac{N_c+1}{N_c}\frac{\alpha(\mu)}{2\pi} \tag{9.3}$$

of the evolution equation (6.7) into the anomalous dimensions (9.1), (9.2) and treat this factor separately. The exact result for the anomalous dimension $\Gamma_{\{n_k=0\}}$ (9.2) agrees with the general structure of the $1/N_c$ expansion (9.1). Comparing the exact result (9.2) with the $1/N_c$ expansion (9.1), we can find the parameter

$$\Delta E = -\frac{1}{4}\,. \tag{9.4}$$

B. Traditional approach to the calculation of anomalous dimensions

The "excitation energies" Ω_k appearing in the large-N_c representation (9.1) for the anomalous dimensions of the baryon operators can be computed by solving Eq. (7.11). Our derivation of this equation was based on the linearized analysis of the asymptotic behavior (7.14) of the solution $W(g,t)$ to the evolution equation (6.13). Let us show how the same result comes from the traditional approach to the calculation of anomalous dimensions.

In the traditional approach one has to diagonalize the effective Hamiltonian corresponding to the evolution equation (6.1)

$$H\tilde{\Psi}_\alpha(x_1,\ldots,x_{N_c}) = \Gamma_\alpha \tilde{\Psi}_\alpha(x_1,\ldots,x_{N_c})\,, \tag{9.5}$$

$$H = \sum_{1\leq i<j\leq N_c} K_{ij}\,. \tag{9.6}$$

Let us show that the eigenvalues Γ_α appearing here coincide with the expression (9.1) at large N_c. First we rewrite Eq. (9.5) in terms of the generating functional for the function $\tilde{\Psi}_\alpha(x_1,\ldots,x_{N_c})$. By analogy with Eq. (5.9) we define

$$\begin{aligned}\tilde{\Phi}_\alpha(g) = N_c^{-N_c/2}\int_0^\infty dy_1\int_0^\infty dy_2\ldots\int_0^\infty dy_{N_c} g_{f_1 s_1}(y_1) g_{f_2 s_2}(y_2)\ldots g_{f_{N_c} s_{N_c}}(y_{N_c})\\ \times\,\tilde{\Psi}_{\alpha,(f_1 s_1)(f_2 s_2)\ldots(f_{N_c} s_{N_c})}\left(\frac{y_1}{N_c},\frac{y_2}{N_c},\ldots,\frac{y_{N_c}}{N_c}\right)\,.\end{aligned} \tag{9.7}$$

Repeating the same steps that were used in the derivation of Eq. (6.7), we find

$$\frac{1}{2}\,(g\otimes g)\cdot K\cdot\left[\left(\frac{\delta}{\delta g}\otimes\frac{\delta}{\delta g}\right)\tilde{\Phi}_\alpha(g)\right] = \Gamma_\alpha\tilde{\Phi}_\alpha(g)\,. \tag{9.8}$$

Next we write the large-N_c ansatz (5.10)

$$\tilde{\Phi}_\alpha(g) = N_c^{\nu_\alpha}\tilde{A}_\alpha(g)\exp\left[N_c W_0(g)\right]\left[1+O\left(N_c^{-1}\right)\right] \tag{9.9}$$

and expand

$$\Gamma_\alpha = N_c E + \Delta\Gamma_\alpha + O\left(N_c^{-1}\right)\,. \tag{9.10}$$

We use the tilded notation $\tilde{\Psi}_\alpha$, $\tilde{\Phi}_\alpha(g)$, $\tilde{A}_\alpha(g)$ in order to distinguish the wave functions and functionals associated with the diagonalization problem (9.5) from the wave functions and functionals Ψ, $\Phi(g)$, $A(g)$ corresponding to the physical baryons states. However, note that the functionals $W_0(g)$, $\xi_k(g)$ appearing in this section are the same as before.

Inserting ansatz (9.9) into Eq. (9.8), we reproduce Eq. (7.2) in the leading order of the $1/N_c$ expansion. In the next order we obtain the equation

$$\begin{aligned}(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta\ln\tilde{A}_\alpha(g)}{\delta g}\right]\\ +\frac{1}{2}\,(g\otimes g)\cdot K\cdot\left[\left(\frac{\delta}{\delta g}\otimes\frac{\delta}{\delta g}\right)W_0(g)\right] = \Delta\Gamma_\alpha\,.\end{aligned} \tag{9.11}$$

In addition to this equation the functionals $\tilde{A}_\alpha(g)$ must be homogeneous,

$$\tilde{A}_\alpha(\lambda g) = \tilde{A}_\alpha(g)\,, \tag{9.12}$$

for any number λ. This condition follows from the property

$$\tilde{\Phi}_\alpha(\lambda g) = \lambda^{N_c}\tilde{\Phi}_\alpha(g) \tag{9.13}$$

and from Eq. (7.4).

Let us use the value $\alpha = 0$ for the ground state. Taking the difference of the equations (9.11) with $\alpha \neq 0$ and with $\alpha = 0$, we find

$$(g \otimes g) \cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta}{\delta g} \ln \frac{\tilde{A}_\alpha(g)}{\tilde{A}_0(g)} \right] = \Delta\Gamma_\alpha - \Delta\Gamma_0 \,. \tag{9.14}$$

Now let us make the ansatz

$$\tilde{A}_\alpha(g) = \tilde{A}_0(g) \prod_k \left[\xi_k(g)\right]^{n_k} \,, \tag{9.15}$$

where $\xi_k(g)$ are solutions of Eq. (7.11). Then

$$(g \otimes g) \cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta}{\delta g} \sum_k n_k \ln \xi_k(g) \right] = \Delta\Gamma_\alpha - \Delta\Gamma_0 \,. \tag{9.16}$$

According to Eq. (7.11) we have

$$(g \otimes g) \cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta}{\delta g} \ln \xi_k(g) \right] = \Omega_k \,. \tag{9.17}$$

Inserting Eq. (9.17) into Eq. (9.16), we find

$$\Delta\Gamma_\alpha - \Delta\Gamma_0 = \sum_k n_k \Omega_k \,. \tag{9.18}$$

Now we combine this with Eq. (9.10):

$$\Gamma_\alpha - \Gamma_0 = \sum_k n_k \Omega_k + O(N_c^{-1}) \,. \tag{9.19}$$

Taking $\alpha = \{n_k\}$, we see that this result agrees with Eq. (9.1) and

$$\Delta\Gamma_0 = \Delta E \,. \tag{9.20}$$

Thus at large N_c the diagonalization of the effective Hamiltonian (9.5) reproduces our old expression for the anomalous dimensions of baryon operators (9.1).

C. Calculation of excitation energies Ω_k

Now we want to find the excitation energies Ω_k. This can be done by solving Eq. (7.11). In this paper we solve this equation in the simplest case of one flavor $N_f = 1$. In order to construct this solution we need auxiliary functionals

$$M_{ks}(g) = \int_0^\infty dy g_s(y) y^{k+1} e^{-X(g)y} \,, \tag{9.21}$$

$$T_{ks}(g) = \frac{M_{ks}(g)}{M_{0s}(g)} \,. \tag{9.22}$$

We shall see that the solutions $\xi_m(g)$ of Eq. (7.11) can be constructed as polynomials

$$\xi_m(g) = P_m[T(g)] \,. \tag{9.23}$$

Obviously we have for any constant λ

$$T_{ks}(\lambda g) = T_{ks}(g) \,. \tag{9.24}$$

Therefore the functionals $\xi_m(g)$ constructed as polynomials of T_{ks} (9.23) will obey condition (7.12).

Assuming the structure (9.23) of the solution, we find

$$\frac{\delta\xi_m(g)}{\delta g_{s_2}(y)} = \sum_{ks} \frac{\partial P_m(T)}{\partial T_{ks}} \frac{\delta T_{ks}(g)}{\delta g_{s_2}(y)}\,,$$

so that Eq. (7.11) can be rewritten in the form

$$\sum_{ks} \frac{\partial P_m(T)}{\partial T_{ks}} \left\{ (g\otimes g)\cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta T_{ks}(g)}{\delta g}\right]\right\} = \Omega_m P_m[T(g)]\,. \tag{9.25}$$

Below we establish an important property of functionals $T_{ks}(g)$:

$$\begin{aligned}&(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta T_{ks}(g)}{\delta g}\right]\\ &=\frac{1}{4}\left[-T_{ks}(g)+\sum_{s_1}\sum_{n=0}^{k} B_{kn}^{s_1 s} T_{ns_1}(g) T_{k-n,s}(g)\right]\,,\end{aligned} \tag{9.26}$$

where $B_{kn}^{s_1 s}$ are some coefficients. The derivation of the equality (9.26) and the calculation of the coefficients $B_{kn}^{s_1 s}$ requires some work [see Eq. (C31) in Appendix C]. But once this work is done, the solution of Eq. (9.25) becomes simple. Inserting Eq. (9.26) into Eq. (9.25), we find

$$\frac{1}{4}\sum_{ks}\frac{\partial P_m(T)}{\partial T_{ks}}\left[-T_{ks}+\sum_{s_1}\sum_{n=0}^{k} B_{kn}^{s_1 s} T_{ns_1} T_{k-n,s}\right] = \Omega_m P_m(T)\,. \tag{9.27}$$

We look for the polynomial solutions (9.23)

$$P_m(T) = \sum_{\{k_i,s_i\}} a^{(m)}_{\{k_i s_i\}} T_{k_1 s_1} T_{k_2 s_2}\dots T_{k_n s_n}\,. \tag{9.28}$$

Let us assign the "weight" k to the functional T_{ks}. Then the product $T_{k_1 s_1} T_{k_2 s_2}\dots T_{k_n s_n}$ will have the weight

$$m = k_1 + k_2 + \ldots + k_n\,. \tag{9.29}$$

Obviously Eq. (9.27) is closed with respect to polynomials $P_m(T)$ of any given weight m. Therefore we can classify the solutions according to their weight (9.29). This is the reason why we identify the notation m for weight (9.29) and the index m of the solution $P_m(T)$. Let us concentrate on the linear terms T_{ms} in $P_m(T)$:

$$P_m(T) = \sum_s a_{ms} T_{ms} + (\text{nonlinear terms})\,. \tag{9.30}$$

Separating these linear terms in Eq. (9.27), we find

$$\frac{1}{4}\sum_s a_{ms}\left[-T_{ms}+\sum_{s_1}\left(B_{mm}^{s_1 s} T_{ms_1} + B_{m0}^{s_1 s} T_{ms}\right)\right] = \Omega_m \sum_s a_{ms} T_{ms}\,. \tag{9.31}$$

We have taken into account the property

$$T_{0s}(g) = 1 \tag{9.32}$$

which is an obvious consequence of Eq. (9.22). The coefficients $B_{mn}^{s_1 s_2}$ depend only on the relative orientation of the helicities s_1, s_2 so that we can use the notation

$$B_{mn}^{ss} = B_{mn}^{+}\,,\quad B_{mn}^{-s,s} = B_{mn}^{-}\,. \tag{9.33}$$

Then

$$\sum_{s_1}\left(B_{mm}^{s_1 s} T_{ms_1} + B_{m0}^{s_1 s} T_{ms}\right) = \left(B_{mm}^{+} + B_{m0}^{+}\right) T_{ms} + \left(B_{mm}^{-} T_{m,-s} + B_{m0}^{-} T_{ms}\right) \tag{9.34}$$

and Eq. (9.31) takes the form

$$\frac{1}{4}\sum_s a_{ms}\left[\left(B^+_{mm}+B^+_{m0}+B^-_{m0}-1\right)T_{ms}+B^-_{mm}T_{m,-s}\right]=\Omega_m\sum_s a_{ms}T_{ms}\,. \tag{9.35}$$

Obviously we have two solutions $a^{\pm}_{ms}$ for a given weight m. One solution is even in s

$$a^+_{ms}=a^+_{m,-s}\,, \tag{9.36}$$

and the other is odd

$$a^-_{ms}=-a^-_{m,-s}\,. \tag{9.37}$$

According to Eqs. (9.23) and (9.30) these two possibilities lead to the solutions

$$\xi^{\pm}_m(g)\equiv P^{\pm}_m[T(g)]=[T_{m+}(g)\pm T_{m-}(g)]+[\text{terms nonlinear in}\,T_{ks}(g)]\,. \tag{9.38}$$

Parameters $\Omega^{\pm}_m$ associated with these solutions can be found from Eq. (9.35):

$$\Omega^{(\pm)}_m=\frac{1}{4}\left(B^+_{mm}\pm B^-_{mm}+B^+_{m0}+B^-_{m0}-1\right)\,. \tag{9.39}$$

The explicit expressions for $B^{\pm}_{mm}$ and $B^{\pm}_{m0}$ can be found in Appendix C [see Eqs. (C36) – (C37)]. As a result, we find

$$\Omega^-_m=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{m+2}+\frac{1}{(m+1)(m+2)}\right]+\sum_{j=2}^{m+1}\frac{1}{j}\,, \tag{9.40}$$

$$\Omega^+_m=\Omega^-_m-\frac{2}{m(m+2)}\,. \tag{9.41}$$

The corresponding solutions of Eq. (9.25) will be denoted $\xi^{\pm}_m(g)$:

$$(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta\xi^{\pm}_m(g)}{\delta g}\right]=\Omega^{\pm}_m\xi^{\pm}_m(g)\,. \tag{9.42}$$

In the above derivation we had $m\geq 1$. However, one should keep in mind that

$$\Omega^+_1=0\,. \tag{9.43}$$

The solution of Eq. (9.25) for this zero mode is

$$\xi^+_1(g)=T_{1,+}(g)+T_{1,-}(g)\,. \tag{9.44}$$

Computing the derivative with respect to X in Eq. (8.16), we find

$$\sum_s T_{1s}(g)=2\,. \tag{9.45}$$

This means that

$$\xi^+_1(g)=2 \tag{9.46}$$

so that Eq. (9.42) for $\xi^+_1(g)$

$$(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta\xi^+_1(g)}{\delta g}\right]=0 \tag{9.47}$$

is satisfied in a trivial way.

In Appendix C we find another zero mode of Eq. (7.11). According to Eq. (C20) we have

$$(g \otimes g) \cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta \xi_{\mathrm{rot}}(g)}{\delta g} \right] = 0\,, \tag{9.48}$$

where

$$\xi_{\mathrm{rot}}(g) = \frac{M_{0+}(g)}{M_{0-}(g)}\,. \tag{9.49}$$

This zero mode corresponds to the spin rotations around the space component of the light-cone vector n^μ used in the definition of the distribution amplitude (5.4), (5.5). The role of this zero mode will be discussed in Sec. IX D.

Let us summarize. The spectrum of the anomalous dimensions of baryon operators is described by Eq. (9.1),

$$\Gamma_{\{n_m^\pm, J_3\}} = \frac{1}{8} N_c + \left(\Delta E + \sum_{m \geq 2} n_m^+ \Omega_m^+ + \sum_{m \geq 1} n_m^- \Omega_m^- \right) + O(N_c^{-1})\,, \tag{9.50}$$

with the parameter ΔE (9.4) and with the excitation energies $\Omega_k^\pm$ (9.40), (9.41).

D. Helicity

Our result for the anomalous dimensions (9.50) is degenerate in helicity J_3. This degeneracy is lifted in the order $O(N_c^{-1})$. It can be seen from the exact (in N_c) result (9.2) for the operators corresponding to $n_m^\pm = 0$. In fact, the conservation of helicity allows us to diagonalize the evolution in terms of generating functionals $\tilde{A}_\alpha(g)$ (9.15) without computing the $O(N_c^{-1})$ corrections to the anomalous dimensions.

Under the axial rotations corresponding to the helicity

$$g'_s = g_s e^{is\phi}\,, \tag{9.51}$$

the functionals $M_{ks}(g)$ (9.21) and $T_{ks}(g)$ (9.22) transform as follows

$$M_{ks}(g') = e^{i\phi s} M_{ks}(g)\,, \tag{9.52}$$

$$T_{ks}(g') = T_{ks}(g)\,. \tag{9.53}$$

The solutions $\xi_m^\pm(g)$ are constructed as functions of $T_{ks}(g)$ [see Eq. (9.23)]. Therefore these solutions are invariant under rotations (9.51)

$$\xi_m^\pm(g') = \xi_m^\pm(g)\,. \tag{9.54}$$

Using Eq. (9.52), we find for the zero mode (9.49)

$$\xi_{\mathrm{rot}}(g') = e^{i\phi} \xi_{\mathrm{rot}}(g)\,. \tag{9.55}$$

In the previous section we showed that the anomalous dimensions of baryon operators diagonalizing the evolution can be parametrized by

$$\alpha = (n_m^\pm, J_3)\,, \tag{9.56}$$

where $n_m^\pm$ are integer numbers and J_3 is the helicity of the baryon operator. Assuming this meaning of the subscript α in the generating functional $\tilde{A}_\alpha$ (9.9)

$$\tilde{A}_\alpha(g) \equiv \tilde{A}_{\{n_m^\pm, J_3\}}(g)\,, \tag{9.57}$$

we can write for the rotations (9.51)

$$\tilde{A}_{\{n_m^\pm, J_3\}}(g') = e^{iJ_3\phi} \tilde{A}_{\{n_m^\pm, J_3\}}(g)\,. \tag{9.58}$$

Now let us rewrite the decomposition (9.15) in the form

$$\tilde{A}_{\{n_m^\pm, J_3\}}(g) = \tilde{A}_0(g) \left\{ \prod_{m \geq 2} \left[\xi_m^+(g) \right]^{n_m^+} \right\} \left\{ \prod_{m \geq 1} \left[\xi_m^-(g) \right]^{n_m^-} \right\} \left[\xi_{\mathrm{rot}}(g) \right]^{J_3}\,. \tag{9.59}$$

Here we have separated the contribution of the zero mode $\xi_{\mathrm{rot}}(g)$ (9.49). Applying transformation (9.51) to the decomposition (9.59) and using Eqs. (9.54), (9.55), and (9.58), we find

$$\tilde{A}_0(g') = \tilde{A}_0(g)\,. \tag{9.60}$$

E. Functional $\tilde{A}_0(g)$

Now we want to compute the functional $\tilde{A}_0(g)$ appearing in the decomposition (9.59). First let us derive an equation for this functional. Taking Eq. (9.11) for the ground state $\alpha = 0$ and using relations (9.4), (9.20), we obtain

$$(g \otimes g) \cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta \ln \tilde{A}_0(g)}{\delta g} \right] + \frac{1}{2} (g \otimes g) \cdot K \cdot \left[\left(\frac{\delta}{\delta g} \otimes \frac{\delta}{\delta g} \right) W_0(g) \right] = -\frac{1}{4} . \tag{9.61}$$

The solution of this equation can be written in terms of functionals T_{ks} (9.22):

$$\tilde{A}_0(g) = \left\{ \sum_s \left\{ T_{2s}(g) - [T_{1s}(g)]^2 \right\} \right\}^{-1/2} . \tag{9.62}$$

A straightforward but tedious calculation [using relations (9.26) and (C8)] shows that this functional obeys equation (9.61). Note that Eq. (9.61) does not fix the solution $\tilde{A}_0(g)$ completely. For example, due to Eq. (9.48) we have the freedom of transformations

$$\tilde{A}_0(g) \to \tilde{A}_0(g) \left[\xi_{\mathrm{rot}}(g) \right]^n . \tag{9.63}$$

The ambiguity is fixed by the additional condition (9.60). Obviously solution (9.62) obeys constraint (9.60) as well as the constraint (9.12).

X. FUNCTIONAL $A_B(g)$ IN THE CASE $N_f = 2$

A. Functional $A_B(g,t)$ in the asymptotic regime $t \to \infty$

In Sec. VIII D we showed how the saddle point method can be used for the calculation of the functional $W_0(g)$ (8.25). The same saddle point method also allows us to compute the pre-exponential factor $A_B(g)$. To this aim we return to Eq. (8.42) and perform the integration over R more carefully, using Eq. (4.22). As a result, we find

$$\Phi_\mu(g) \overset{\mu\to\infty}{=} \frac{a^{(B)} c_{TN_c} N_c}{2\pi} \frac{2^{N_c+2}}{\sqrt{2\pi N_c^3}} \exp\left[-N_c \sigma(\mu)\right] \times \int dZ D^T_{J_3 T_3}\left(R_0^{-1}\right) \exp\left\{ N_c \left\{ -iZ + \frac{1}{2} \ln\det\left[\int_0^\infty dy g(y) y e^{iZy} \right] \right\} \right\} , \tag{10.1}$$

where

$$R_0^{-1} = \frac{g_{\mathrm{eff}}^{\mathrm{tr}}}{\sqrt{\det g_{\mathrm{eff}}}} , \tag{10.2}$$

$$g_{\mathrm{eff}} = \int_0^\infty dy g(y) y e^{iZy} . \tag{10.3}$$

Next we perform the saddle point integration over Z which yields

$$\Phi_\mu(g) \overset{\mu\to\infty}{=} \frac{2^{N_c+1}\sqrt{2} a^{(B)} c_{TN_c}}{\pi N_c} \exp\left[-N_c \sigma(\mu)\right] D^T_{J_3 T_3}(R_0^{-1}) \times \left\{ -\frac{\partial^2}{\partial Z^2} \ln\det\left[\int_0^\infty dy g(y) y e^{iZy} \right] \right\}^{-1/2} \exp\left\{ N_c \left\{ -iZ + \frac{1}{2} \ln\det\left[\int_0^\infty dy g(y) y e^{iZy} \right] \right\} \right\} , \tag{10.4}$$

where Z is determined by the saddle point equation (8.46). It is convenient to use the variable $X = -iZ$ (8.47) instead of Z. Introducing the compact notation

$$L_k = \int_0^\infty dy g(y) y^{k+1} e^{-Xy} , \tag{10.5}$$

we can rewrite Eq. (10.2) in the form

$$R_0^{-1} = \frac{L_0^{\mathrm{tr}}}{\sqrt{\det L_0}} \,. \tag{10.6}$$

Using notation (10.5), we find

$$\frac{\partial}{\partial Z}\left\{-iZ + \frac{1}{2}\ln\det\left[\int_0^\infty dy g(y) y e^{iZy}\right]\right\} = i\left[-1 + \frac{1}{2}\mathrm{Tr}\left(L_0^{-1}L_1\right)\right] . \tag{10.7}$$

Applying the identity

$$\frac{\partial}{\partial Z}\mathrm{Tr}\left(L_0^{-1}L_1\right) = i\mathrm{Tr}\left[L_0^{-1}L_2 - \left(L_0^{-1}L_1\right)^2\right] , \tag{10.8}$$

we derive from Eq. (10.7)

$$\frac{\partial^2}{\partial Z^2}\left\{\ln\det\left[\int_0^\infty dy g(y) y e^{iZy}\right]\right\} = -\mathrm{Tr}\left[L_0^{-1}L_2 - \left(L_0^{-1}L_1\right)^2\right] . \tag{10.9}$$

Inserting Eq. (10.7) into the saddle point equation (8.46), we find

$$\mathrm{Tr}\left(L_0^{-1}L_1\right) = 2 \,. \tag{10.10}$$

This saddle point equation implicitly defines the dependence of X on g

$$X = X(g) \,. \tag{10.11}$$

Inserting this dependence $X(g)$ into Eq. (10.5) we see that L_k also become functionals of g only

$$L_k = L_k(g) \,. \tag{10.12}$$

Using Eqs. (10.5), (10.6), and (10.9), we find from Eq. (10.4)

$$\Phi_\mu(g) \overset{\mu\to\infty}{=} \frac{2^{N_c+1}\sqrt{2}a^{(B)}c_{TN_c}}{\pi N_c}\exp\left[-N_c\sigma(\mu)\right] D^T_{J_3T_3}\left(\frac{L_0^{\mathrm{tr}}}{\sqrt{\det L_0}}\right)$$
$$\times\left\{\mathrm{Tr}\left[L_0^{-1}L_2 - \left(L_0^{-1}L_1\right)^2\right]\right\}^{-1/2}\exp\left[N_c\left(-iZ + \frac{1}{2}\ln\det L_0\right)\right] . \tag{10.13}$$

Now we insert expression (4.8) for c_{TN_c} and apply the property of Wigner functions (A3):

$$\Phi_\mu(g) \overset{\mu\to\infty}{=} a^{(B)}2^{N_c/2}\left(\frac{8}{\pi^3 N_c}\right)^{1/4}\exp\left[-N_c\sigma(\mu)\right]\sqrt{2T+1}D^T_{T_3J_3}\left(\frac{L_0}{\sqrt{\det L_0}}\right)$$
$$\times\left\{\mathrm{Tr}\left[L_0^{-1}L_2 - \left(L_0^{-1}L_1\right)^2\right]\right\}^{-1/2}\exp\left[N_c\left(-iZ + \frac{1}{2}\ln\det L_0\right)\right] . \tag{10.14}$$

In the leading order of the $1/N_c$ expansion, $\sigma(\mu)$ is given by expression (8.39). We must also take into account the $O(N_c^0)$ correction ΔE to the energy N_cE. Then

$$\Phi(g,t) \overset{t\to\infty}{=} a^{(B)}N_c^{-1}2^{N_c/2}\left(\frac{8}{\pi^3 N_c}\right)^{1/4}\exp\left[-\left(N_cE + \Delta E\right)t\right]\sqrt{2T+1}D^T_{T_3J_3}\left(\frac{L_0}{\sqrt{\det L_0}}\right)$$
$$\times\left\{\mathrm{Tr}\left[L_0^{-1}L_2 - \left(L_0^{-1}L_1\right)^2\right]\right\}^{-1/2}\exp\left\{N_c\left[X(g) + \frac{1}{2}\ln\det L_0\right]\right\} . \tag{10.15}$$

This leads to the old result (8.28) for the functional $W_{\mathrm{as}}(g,t)$

$$W_{\mathrm{as}}(g,t) = W_0(g) - Et \,, \tag{10.16}$$

$$W_0(g) = X(g) + \frac{1}{2}\ln\left[\det L_0(g)\right] \tag{10.17}$$

but now we also know the functional $A_B(g,t)$

$$A_B^{T,T_3J_3}(g,t) = c_B\sqrt{2T+1}D^T_{T_3J_3}\left[Q_0(g)\right]A_0(g,t)\;, \tag{10.18}$$

where c_B is a constant independent of g and

$$Q_0(g) = \frac{L_0(g)}{\sqrt{\det L_0(g)}}\,, \tag{10.19}$$

$$A_0(g,t) = \exp\left(-t\Delta E\right)\left\{\mathrm{Tr}\left\{L_0^{-1}(g)L_2(g) - \left[L_0^{-1}(g)L_1(g)\right]^2\right\}\right\}^{-1/2}\;. \tag{10.20}$$

Note that the functional $A_B(g,t)$ depends on the type of the state B. In the asymptotic regime $t\to\infty$ this dependence appears in Eq. (10.18) only via the quantum numbers TJ_3T_3 and via the constant c_B. The origin of this simple dependence is obvious: the asymptotic distribution amplitude is controlled by the single operator with the lowest anomalous dimension corresponding to given TJ_3T_3. The result (10.18) can be used only for baryons obeying condition $|J_3|\leq T$ (8.29), because the underlying expression (8.30) for the asymptotic distribution amplitude was valid only for this case. For the lowest $O(N_c^{-1})$ baryon excitations with $J=T$ (3.1), the condition $|J_3|\leq T$ is satisfied automatically.

B. Comments on the noncommutativity of limits $N_c\to\infty$ and $t\to\infty$

In the previous section we have computed the functional $A_B(g,t)$ in the asymptotic regime. This calculation was based on the analysis of the double limit $N_c\to\infty$ and $t\to\infty$. As was discussed in Sec. VII, these two limits do not commute. Therefore our results (10.18) – (10.20) must be taken with a certain care. The calculation of $A_B(g,t)$ made in Sec. X A relied on the results of Sec. VIII C, which were based on the explicit expression for the asymptotic distribution amplitude. Therefore the work of Sec. X A corresponded to the case when the asymptotic limit $t\to\infty$ precedes the large-N_c limit: the asymptotic limit selects the contribution corresponding to the lowest anomalous dimension (depending on the quantum numbers JJ_3TT_3), and only after that we use the large-N_c asymptotic distribution amplitude corresponding to these chosen quantum numbers JJ_3TT_3. Strictly speaking, the calculation of Sec. X A should be interpreted as a calculation of the $N_f=2$ analog of the functional $\tilde{A}_\alpha(g)$ (9.9), which appears in the problem of the diagonalization of the anomalous dimensions at large N_c.

Although the *method* standing behind the calculation in Sec. X A is based on the limit

$$\text{first}\,t\to\infty,\ \text{then}\,N_c\to\infty\,, \tag{10.21}$$

the *form* of presentation used in Sec. X A corresponds to the limit

$$\text{first}\,N_c\to\infty,\ \text{then}\,t\to\infty\,. \tag{10.22}$$

Indeed, the functional $A_B(g,t)$ is defined only in the limit $N_c\to\infty$. Before investigating the large-t behavior of $A_B(g,t)$ we must take the large-N_c limit in order to define this functional.

To summarize, the results of the previous section were derived for the regime (10.21). The possibility of the extrapolation of these results to the region (10.22) requires an additional investigation.

If we stay in the "safe" regime (10.21) corresponding to the problem of the diagonalization of the anomalous dimensions, then we can rewrite Eq. (10.18) in the form similar to the $N_f=1$ equation (9.59):

$$\tilde{A}_{TT_3J_3}(g) = \sqrt{2T+1}D^T_{T_3J_3}\left[Q_0(g)\right]\tilde{A}_0(g)\,, \tag{10.23}$$

where

$$\tilde{A}_0(g) = \left\{\mathrm{Tr}\left\{L_0^{-1}(g)L_2(g) - \left[L_0^{-1}(g)L_1(g)\right]^2\right\}\right\}^{-1/2}\;. \tag{10.24}$$

The zero mode factor $\sqrt{2T+1}D^T_{T_3J_3}\left[Q_0(g)\right]$ in Eq. (10.23) plays the same role as the $N_f=1$ zero mode factor $\left[\xi_{\mathrm{rot}}(g)\right]^{J_3}$ in Eq. (9.59). Equation (10.23) contains no analogs of the nonzero mode factors $\xi_m^{\pm}(g)$ of Eq. (9.59) because the representation (10.23) was derived for the asymptotic wave function corresponding to the lowest anomalous dimension (for given TT_3J_3).

The analogy between Eq. (10.23) and the $N_f = 1$ equations of Sec. IX C can be continued. For example, the $N_f = 2$ zero mode $Q_0(g)$ obeys the same equation (9.48) as the $N_f = 1$ zero mode $\xi_{\rm rot}(g)$:

$$(g \otimes g) \cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta Q_0(g)}{\delta g} \right] = 0\,. \tag{10.25}$$

The check of this equation is straightforward but tedious. One has to use expressions (10.17), (10.19) for the functionals $W_0(g)$, $Q_0(g)$ via functionals $L_k(g)$ and the identity (10.10) for $L_k(g)$.

C. Zero modes and the structure of the functional $A_B(g)$

In Sec. III D we have suggested the factorized form (3.8) for the functional $A_B(g)$. However, this naive factorized form is modified by zero modes. The precise form of this modification depends on the involved symmetries. For example, in the problem of the asymptotic distribution amplitude for $N_f = 1$ flavor we have only one nontrivial zero mode $\xi_{\rm rot}(g)$ (9.49). This zero mode corresponds to the Abelian symmetry associated with the helicityJ_3. Therefore the zero mode factor $[\xi_{\rm rot}(g)]^{J_3}$ appearing in Eq. (9.59) does not violate the naive factorized form (3.8) of the functional $A_B(g)$.

The case of $N_f = 2$ quark flavors is more complicated. The results obtained in this paper give only indirect information about the functional $A_B(g)$. We know the functional $A_B(g)$

1) in the toy quark model where it is given by expression (4.12) for the baryons with $T = J$,

2) in the asymptotic $t \to \infty$ case described by Eq. (10.23), which was derived for baryons with the helicity J_3 constrained by the condition $|J_3| \leq T$.

Comparing Eqs. (4.12) and (10.23), we see that in both cases the functional $A_B(g)$ has the form

$$A_{T=J,T_3J_3}(g) = \text{const}\, D^{T=J}_{T_3J_3}\left[Q(g)\right] A_0(g)\,. \tag{10.26}$$

In the toy quark model we have

$$Q(g) = \frac{g}{\sqrt{\det g}}\,, \quad A_0(g) = 1 \quad (\text{toy quark model})\,, \tag{10.27}$$

whereas in the case of the asymptotic wave function ($t \to \infty$)

$$Q(g) = \frac{L_0(g)}{\sqrt{\det L_0(g)}}\,, \quad A_0(g) = \left\{ \mathrm{Tr} \left\{ L_0^{-1}(g) L_2(g) - \left[L_0^{-1}(g) L_1(g) \right]^2 \right\} \right\}^{-1/2} \quad (\text{asymptotic regime})\,. \tag{10.28}$$

In both cases $Q(g)$ is an $SL(2, C)$ matrix.

Comparing the results obtained in the naive quark model (10.27) and in the asymptotic limit (10.28), we can make the conjecture that in the nonasymptotic regime the true QCD functional $A_{TT_3J_3}(g, t)$ for the lowest $O(N_c^{-1})$ excited baryons with $J = T$ will have the same structure as Eq. (10.26)

$$A_{T=J,T_3J_3}(g, t) = \text{const} D^{T=J}_{T_3J_3}\left[Q(g, t)\right] A_0(g, t)\,, \tag{10.29}$$

$$\det Q(g, t) = 1\,, \tag{10.30}$$

where $Q(g, t)$ is some unknown functional of g with values in $SL(2, C)$ which is determined by the dynamics of large-N_c QCD.

Let us show that the conjecture (10.29) and the $SL(2, C)$ constraint (10.30) are compatible with the evolution equation. Applying the general evolution equation (6.20) to the expression (10.29), we see that it takes the form

$$\frac{\partial}{\partial t} \frac{D^{T'=J'}_{T_3'J_3'}\left[Q(g, t)\right]}{D^{T=J}_{T_3J_3}\left[Q(g, t)\right]} = -\left\{ (g \otimes g) \cdot K \cdot \left[\frac{\delta W(g, t)}{\delta g} \otimes \frac{\delta}{\delta g} \frac{D^{T'=J'}_{T_3'J_3'}\left[Q(g, t)\right]}{D^{T=J}_{T_3J_3}\left[Q(g, t)\right]} \right] \right\}\,. \tag{10.31}$$

Let us show that this equation will hold automatically for any $T = J, T_3, J_3$ and $T' = J', T_3', J_3'$ if the functional $Q(g, t)$ obeys the evolution equation

$$\frac{\partial}{\partial t} Q_{fs}(g, t) = -\left\{ (g \otimes g) \cdot K \cdot \left[\frac{\delta W(g, t)}{\delta g} \otimes \frac{\delta Q_{fs}(g, t)}{\delta g} \right] \right\}\,. \tag{10.32}$$

Indeed, Eq. (10.32) has an obvious property: If $Q(g,t)$ obeys this equation, then any function $F(Q)$ depending on the matrix elements Q_{fs} will obey the same equation:

$$\frac{\partial}{\partial t} F\left[Q(g,t)\right] = -\left\{ (g\otimes g)\cdot K \cdot \left[\frac{\delta W(g,t)}{\delta g} \otimes \frac{\delta F\left[Q(g,t)\right]}{\delta g} \right] \right\} . \tag{10.33}$$

Applying this property to the function

$$F(Q) = \frac{D^{T'=J'}_{T'_3 J'_3}(Q)}{D^{T=J}_{T_3 J_3}(Q)} , \tag{10.34}$$

we derive equation (10.31). If we take

$$F(Q) = \det Q , \tag{10.35}$$

then we find from Eq. (10.33)

$$\frac{\partial}{\partial t} \det Q(g,t) = -\left\{ (g\otimes g)\cdot K \cdot \left[\frac{\delta W(g,t)}{\delta g} \otimes \frac{\delta \det Q(g,t)}{\delta g} \right] \right\} . \tag{10.36}$$

Obviously this equation is consistent with the $SL(2,C)$ constraint (10.30).

Certainly the consistency of the conjecture (10.29), (10.30) with the evolution equation and with the asymptotic case cannot replace the proof of this conjecture. In Ref. [24] we present additional arguments in favor of Eq. (10.29) based on the large-N_c spin-flavor symmetry [9, 10, 21, 22, 23] and on the soft-pion theorem for the baryon distribution amplitude [49]. In Ref. [24] we also derive the correct version of the naive factorized ansatz (3.8) for the $O(N_c^0)$ excited states, taking into account the modification of Eq. (3.8) due to the zero modes.

XI. CONCLUSIONS

This paper shows how the systematic $1/N_c$ expansion can be applied to the baryon wave function. The main idea of the method is to introduce a generating functional for the baryon wave function and to construct the $1/N_c$ expansion for this generating functional rather than for the baryon wave function itself.

A specific feature of this functional is its exponential dependence on N_c. The exponential part of this functional is universal for all low-lying baryons [including the $O(N_c^{-1})$ and $O(N_c^0)$ excited resonances and meson-baryon scattering states]. The pre-exponential factors depend on the baryon (baryon-meson) state but these factors have a simple representation in terms of functionals associated with elementary excitations (or single meson scattering states).

The developed formalism was applied to the light-cone baryon wave function (distribution amplitude). It is shown that the structure of the $1/N_c$ expansion is compatible with the evolution equation. A nonlinear evolution equation is derived for the leading (exponential) term of the generating functional for the baryon distribution amplitude. The nonlinearity of this equation can be understood in terms of the well-known analogy between the large-N_c limit and the semiclassical approximation. The usual linear evolution equation is an analog of the quantum nonstationary Schrödinger equation, whereas the nonlinear evolution equation appearing in the large-N_c limit corresponds to the classical Hamilton-Jacobi equation.

The nonlinear evolution equation derived in this paper was successfully used for the diagonalization of the anomalous dimensions of baryon operators of the leading twist. This problem was reduced to the linearized analysis of perturbations of the asymptotic solution of the Hamilton-Jacobi equation controlling the evolution in the large N_c limit. Although the nonlinear evolution equation has a rather nontrivial functional form, this problem was solved analytically, and simple expressions were found for the anomalous dimensions of baryon operators in two orders of the $1/N_c$ expansion.

Acknowledgments

I am grateful to Ya. I. Azimov, G. Kirilin, N. Kivel, A. Manashov, V. Yu. Petrov, M. V. Polyakov and N. Stefanis for interesting discussions. This work was supported by DFG and BMBF.

APPENDIX A: BARYON WAVE FUNCTION IN THE TOY QUARK MODEL

1. Wigner D functions

The quark wave function (4.2) is represented as an integral over the $SU(2)$ group containing Wigner functions $D^j_{m_1m_2}(R)$ with the well known properties

$$D^j_{m_1m_2}(R_1R_2) = \sum_{m=-j}^{j} D^j_{m_1m}(R_1)D^j_{mm_2}(R_2), \tag{A1}$$

$$\int dR \left[D^j_{m_1m_2}(R)\right]^* D^{j'}_{m'_1m'_2}(R) = \frac{1}{2j+1}\delta_{jj'}\delta_{m_1m'_1}\delta_{m_2m'_2}\,. \tag{A2}$$

For the transposed matrix R^{tr} we have

$$D^j_{m_1m_2}(R^{\mathrm{tr}}) = D^j_{m_2m_1}(R)\,. \tag{A3}$$

The characters of the irreducible $SU(2)$ representations

$$\chi_j(R) = \sum_{m=-j}^{j} D^j_{mm}(R) \tag{A4}$$

have the properties

$$\chi_j(R) = \chi^*_j(R) = \chi_j(R^{-1})\,, \tag{A5}$$

$$\chi_{1/2}(R) = \mathrm{Tr}R\,, \tag{A6}$$

$$\int dR\chi^*_{j_1}(R)\chi_{j_2}(R) = \delta_{j_1j_2}\,. \tag{A7}$$

2. Normalization integral

Let us compute the constant c_{TN_c} appearing in Eq. (4.2). This constant c_{TN_c} is fixed by the normalization condition (4.4):

$$|c_{TN_c}|^2 \int dR' \int dR D^T_{J_3T_3}(R^{-1}) \left[D^T_{J_3T_3}(R'^{-1})\right]^* \left[\mathrm{Tr}\left(RR'^{+}\right)\right]^{N_c} = 1\,. \tag{A8}$$

Changing the integration variable $R \to RR'$ and using the properties of the D functions (A1) – (A2), we can simplify this integral:

$$\int dR' \int dR D^T_{J_3T_3}(R^{-1}) \left[D^T_{J_3T_3}(R'^{-1})\right]^* \left[\mathrm{Tr}\left(RR'^{+}\right)\right]^{N_c}$$
$$= \frac{1}{(2T+1)^2}\int dR \sum_m D^T_{mm}(R^{-1})\,(\mathrm{Tr}R)^{N_c}\,. \tag{A9}$$

According to Eqs. (A4) – (A6) we have

$$\sum_m D^T_{mm}(R^{-1})\,(\mathrm{Tr}R)^{N_c} = \chi^*_T(R)\left[\chi_{1/2}(R)\right]^{N_c}\,. \tag{A10}$$

Now we derive from Eqs. (A8) – (A10)

$$c_{TN_c} = \frac{2T+1}{\sqrt{I_{N_c,T}}}\,, \tag{A11}$$

where

$$I_{N_c,j} \equiv \int dR\chi_j^*(R)\,(\mathrm{Tr}R)^{N_c} = \int dR\chi_j^*(R)\left[\chi_{1/2}(R)\right]^{N_c}\,. \tag{A12}$$

This integral has a simple meaning. It counts how many times the j representation appears in the decomposition of the tensor product of N_c spin-$\frac{1}{2}$ representations. Combining this group interpretation of the $I_{N_c,j}$ with the standard rules of the spin addition, we arrive at the recursion relation

$$I_{j,N_c+1} = I_{j+\frac{1}{2},N_c} + I_{j-\frac{1}{2},N_c} \tag{A13}$$

with the initial conditions

$$I_{\frac{1}{2},1} = 1, \quad I_{-\frac{1}{2},N_c} = 0\,. \tag{A14}$$

It is easy to check that the solution of Eqs. (A13), (A14) is

$$I_{j,N_c} = \frac{N_c!(2j+1)}{\left(\frac{N_c}{2}+j+1\right)!\left(\frac{N_c}{2}-j\right)!}\,. \tag{A15}$$

Inserting this result into Eq. (A11), we find

$$c_{TN_c} = \sqrt{\frac{2T+1}{N_c!}\left(\frac{N_c}{2}+T+1\right)!\left(\frac{N_c}{2}-T\right)!}\,. \tag{A16}$$

3. Function $\phi_{TT_3J_3}(g)$

According to Eq. (4.6)

$$\phi_{TT_3J_3}(g) = c_{TN_c}\int dR D^T_{J_3T_3}(R^{-1})\left[\mathrm{Tr}\left(Rg^{\mathrm{tr}}\right)\right]^{N_c}\,. \tag{A17}$$

In the case when g is an $SU(2)$ matrix we can compute this integral changing the variable $R \to R\left(g^{\mathrm{tr}}\right)^{-1}$

$$\begin{aligned}\phi_{TT_3J_3}(g) &= c_{TN_c}\int dR D^T_{J_3T_3}(g^{\mathrm{tr}}R^{-1})\,(\mathrm{Tr}R)^{N_c} \\ &= c_{TN_c}\sum_{T_3'=-T}^{T}\int dR D^T_{J_3T_3'}(g^{\mathrm{tr}})D^T_{T_3'T_3}(R^{-1})\,(\mathrm{Tr}R)^{N_c}\,.\end{aligned} \tag{A18}$$

Using the characters (A4) – (A6), we can express this integral in terms of I_{j,N_c} (A12)

$$\begin{aligned}\int dR D^T_{T_3'T_3}(R^{-1})\,(\mathrm{Tr}R)^{N_c} &= \frac{\delta_{T_3'T_3}}{2T+1}\int dR\sum_{m=-T}^{T}D^T_{mm}(R^{-1})\,(\mathrm{Tr}R)^{N_c} \\ &= \frac{\delta_{T_3'T_3}}{2T+1}\int dR\chi_T(R)\,(\mathrm{Tr}R)^{N_c} = \frac{\delta_{T_3'T_3}}{2T+1}I_{T,N_c}\,.\end{aligned} \tag{A19}$$

Thus

$$\phi_{TT_3J_3}(g) = D^T_{J_3T_3}(g^{\mathrm{tr}})\frac{c_{TN_c}I_{T,N_c}}{2T+1}\,. \tag{A20}$$

Using the above result for c_{TN_c} (A11) and the property of Wigner functions (A3), we find

$$\phi_{TT_3J_3}(g) = \sqrt{I_{T,N_c}}D^T_{T_3J_3}(g)\,. \tag{A21}$$

This result is derived for $SU(2)$ matrices g. By definition (4.6), $\phi_{TT_3J_3}(g)$ is an analytical function of g. Therefore we can "analytically continue" the equality (A21) to arbitrary $SL(2,C)$ matrices g [assuming the $SL(2,C)$ version of the Wigner function $D^T_{T_3J_3}(g)$].

Now let us consider arbitrary matrices g. According to the definition of $\phi_{TT_3J_3}(g)$ (4.6) we have

$$\phi_{TT_3J_3}(\lambda g) = \lambda^{N_c}\phi_{TT_3J_3}(g)\,. \tag{A22}$$

Therefore

$$\phi_{TT_3J_3}(g) = (\det g)^{N_c/2}\,\phi_{TT_3J_3}\left[g\,(\det g)^{-1/2}\right]\,. \tag{A23}$$

Since $g\,(\det g)^{-1/2}$ is an $SL(2,C)$ matrix, we can use the result (A21)

$$\phi_{TT_3J_3}\left[g\,(\det g)^{-1/2}\right] = \sqrt{I_{T,N_c}}\,D^T_{T_3J_3}\left[g\,(\det g)^{-1/2}\right]\,. \tag{A24}$$

Combining Eqs. (A23), (A24) and using the expression (A11), we find

$$\phi_{TT_3J_3}(g) = \frac{2T+1}{c_{TN_c}}\,(\det g)^{N_c/2}\,D^T_{T_3J_3}\left[g\,(\det g)^{-1/2}\right]\,. \tag{A25}$$

APPENDIX B: EVOLUTION KERNEL

A detailed discussion of the evolution equation, kernels and anomalous dimensions for the baryon distribution amplitude can be found in the original paper [6] and in later publications, e.g. [35, 36]. Here we list only those relations which are used in this paper.

1. N_c dependence

The leading-order evolution equations for the quark distribution amplitudes of mesons

$$\mu\frac{\partial}{\partial\mu}\Psi^\mu_{\mathrm{meson}}(x_1,x_2) = -\frac{N_c^2-1}{2N_c}\frac{\alpha_s(\mu)}{\pi}K_{12}\Psi^\mu_{\mathrm{meson}}(x_1,x_2) \tag{B1}$$

and baryons

$$\mu\frac{\partial}{\partial\mu}\Psi^\mu_{\mathrm{baryon}}(x_1,\ldots,x_{N_c}) = -\frac{N_c+1}{2N_c}\frac{\alpha_s(\mu)}{\pi}\sum_{1\le i<j\le N_c}K_{ij}\Psi^\mu_{\mathrm{baryon}}(x_1,\ldots,x_{N_c}) \tag{B2}$$

contain the same "pair interactions" K_{ij}. The operator K_{ij} acts on the variables x_i, x_j. The N_c-dependent factors in Eqs. (B1) and (B2) originate from the contraction of the color structure of the one-gluon exchange

$$\sum_{a=1}^{N_c^2-1} t^a_{c_1c_1'}t^a_{c_2c_2'} = \frac{1}{2}\left(\delta_{c_1c_2'}\delta_{c_2c_1'} - \frac{1}{N_c}\delta_{c_1c_1'}\delta_{c_2c_2'}\right)\,, \tag{B3}$$

$$\mathrm{Sp}\left(t^at^b\right) = \frac{1}{2}\delta^{ab} \tag{B4}$$

with the color singlet projectors corresponding to the hadron states. This gives for mesons

$$\left(\sum_{a=1}^{N_c^2-1} t^a_{c_1c_1'}t^a_{c_2c_2'}\right)\delta_{c_1c_2'} = \frac{N_c^2-1}{2N_c}\delta_{c_1'c_2} \tag{B5}$$

and for baryons

$$\left(\sum_{a=1}^{N_c^2-1} t^a_{c_1c_1'}t^a_{c_2c_2'}\right)\varepsilon_{c_1'c_2'c_3\ldots c_{N_c}} = -\frac{N_c+1}{2N_c}\varepsilon_{c_1c_2c_3\ldots c_{N_c}}\,. \tag{B6}$$

2. Properties of the evolution kernel

The evolution equations (B1) and (B2) are written in a compact form in terms of operators K_{ij}. The action of K_{12} is described by the kernel $\tilde{K}^{s_1 s_2}(x_1, x_2; y_1, y_2)$ introduced in Eq. (6.4). The spin structure of this kernel is

$$\tilde{K}^{s_1 s_2}(x_1, x_2; y_1, y_2) = \delta^{s_1 s_2} K^{+}(x_1, x_2; y_1, y_2) + \delta^{s_1, -s_2} K^{-}(x_1, x_2; y_1, y_2)\,. \tag{B7}$$

We assume that the momentum conserving delta function is included into the definition of the evolution kernel:

$$\tilde{K}^{s_1 s_2}(x_1, x_2; y_1, y_2) \sim \delta(x_1 + x_2 - y_1 - y_2)\,. \tag{B8}$$

One can introduce homogeneous polynomials [27, 35]

$$R_{n+1}(x_1, x_2) \;=\; \sum_{k=0}^{n} \frac{n!(n+2)!(-1)^k}{k!(k+1)!(n-k)!(n-k+1)!}\, x_1^k x_2^{n-k}\,, \tag{B9}$$

which can be expressed in terms of Gegenbauer polynomials $C_n^{3/2}$:

$$R_{n+1}(x_1, x_2) = \frac{2}{n+1}(x_1 + x_2)^n C_n^{3/2}\left(\frac{x_2 - x_1}{x_1 + x_2}\right)\,, \tag{B10}$$

The polynomials $R_{n+1}(x_1, x_2)$ diagonalize the evolution kernel:

$$\int_0^\infty dy_1 \int_0^\infty dy_2 \tilde{K}^{s_1 s_2}(x_1, x_2; y_1, y_2) R_{n+1}(y_1, y_2) y_1 y_2 = \frac{1}{2}\gamma_n^{s_1 s_2} R_{n+1}(x_1, x_2) x_1 x_2\,. \tag{B11}$$

The anomalous dimensions $\gamma_n^{s_1 s_2}$ are

$$\gamma_n^{s_1 s_2} = 1 + 4\sum_{k=2}^{n+1}\frac{1}{k} - \frac{2\delta_{s_1, -s_2}}{(n+1)(n+2)}\,. \tag{B12}$$

We have

$$\int_0^\infty dy_1 \int_0^\infty dy_2 K^{\pm}(x_1, x_2; y_1, y_2) R_{n+1}(y_1, y_2) y_1 y_2 = \frac{1}{2}\gamma_n^{\pm} R_{n+1}(x_1, x_2) x_1 x_2\,. \tag{B13}$$

Here

$$\gamma_n^{s_1 s_2} = \delta^{s_1 s_2}\gamma_n^{+} + \delta^{s_1, -s_2}\gamma_n^{-}\,, \tag{B14}$$

$$\gamma_n^{+} = 1 + 4\sum_{k=2}^{n+1}\frac{1}{k}\,, \tag{B15}$$

$$\gamma_n^{-} = 1 + 4\sum_{k=2}^{n+1}\frac{1}{k} - \frac{2}{(n+1)(n+2)}\,. \tag{B16}$$

Since the kernel $K^{\pm}(x_1, x_2; y_1, y_2)$ contains the delta function (B8), we immediately conclude that

$$\int_0^\infty dy_1 \int_0^\infty dy_2 K^{\pm}(x_1, x_2; y_1, y_2) R_{n+1}(y_1, y_2) y_1 y_2 f(y_1 + y_2) = \frac{1}{2}\gamma_n^{\pm} R_{n+1}(x_1, x_2) x_1 x_2 f(x_1 + x_2) \tag{B17}$$

for any function f.

In particular,

$$\begin{aligned}&\int_0^\infty dy_1 \int_0^\infty dy_2 K^{\pm}(x_1, x_2; y_1, y_2) R_{n+1}(y_1, y_2) y_1 y_2 \delta(y_1 + y_2 - \lambda)\\ &= \frac{1}{2}\gamma_n^{\pm} R_{n+1}(x_1, x_2) x_1 x_2 \delta(x_1 + x_2 - \lambda)\,,\end{aligned} \tag{B18}$$

$$\int_0^\infty dy_1 \int_0^\infty dy_2 K^{\pm}(x_1,x_2;y_1,y_2) R_{n+1}(y_1,y_2) y_1 y_2 e^{-a(y_1+y_2)} = \frac{1}{2}\gamma_n^{\pm} R_{n+1}(x_1,x_2) x_1 x_2 e^{-a(x_1+x_2)} \,. \tag{B19}$$

Taking $n=0$ in these equations, we obtain

$$\int_0^\infty dy_1 \int_0^\infty dy_2 \tilde{K}^{s_1 s_2}(x_1,x_2;y_1,y_2) y_1 y_2 \delta(y_1+y_2-\lambda) = \frac{1}{2}\delta^{s_1 s_2} x_1 x_2 \delta(x_1+x_2-\lambda) \,, \tag{B20}$$

$$\int_0^\infty dy_1 \int_0^\infty dy_2 \tilde{K}^{s_1 s_2}(x_1,x_2;y_1,y_2) y_1 y_2 e^{-a(y_1+y_2)} = \frac{1}{2}\delta^{s_1 s_2} x_1 x_2 e^{-a(x_1+x_2)} \,. \tag{B21}$$

The evolution kernel can be decomposed in Gegenbauer polynomials

$$\begin{aligned} &4\sum_{n=0}^{\infty} \frac{\gamma_n^{\varepsilon}}{h_n} C_n^{3/2}(2y-1) C_n^{3/2}(2x-1) \\ &= -\left(\frac{1}{|x-y|} + \delta_{\varepsilon,-}\right)\left[\frac{\theta(y-x)}{(1-x)y} + \frac{\theta(x-y)}{(1-y)x}\right] \quad (x \neq y)\,, \end{aligned} \tag{B22}$$

where

$$h_n = \frac{(n+2)(n+1)}{n+\frac{3}{2}} \tag{B23}$$

is the normalization constant for Gegenbauer polynomials:

$$\int_{-1}^{1} dz (1-z^2) C_m^{3/2}(z)\, C_n^{3/2}(z) = h_m \delta_{mn} \,. \tag{B24}$$

Special care is needed for the singularities appearing at $x=y$ in Eq. (B22). The precise integral form of this equality is

$$\begin{aligned} &4\sum_{n=0}^{\infty} \frac{\gamma_n^{\varepsilon}}{h_n} C_n^{3/2}(2y-1) \int_0^1 dx\, C_n^{3/2}(2x-1)\, x(1-x)\phi(x) \\ &= -\int_0^1 dx \left[\delta_{\varepsilon,-}\phi(x) + \frac{\phi(x)-\phi(y)}{|x-y|}\right]\left[\frac{x}{y}\theta(y-x) + \frac{1-x}{1-y}\theta(x-y)\right] + \frac{1}{2}\phi(y)\,, \end{aligned} \tag{B25}$$

where $\phi(x)$ is an arbitrary function.

APPENDIX C: FUNCTIONALS $T_{ks}(g)$

In this appendix we study the properties of functionals $T_{ks}(g)$. These functionals are defined by Eq. (9.22). They play an important role in Sec. IX C where we solve Eq. (7.11).

1. Variational derivatives

Let us start from the variational derivative of the functional $M_{ks}(g)$ (9.21):

$$\frac{\delta}{\delta g_s(y)} M_{ks'}(g) = \delta_{ss'} y^{k+1} e^{-X(g)y} - M_{k+1,s'}(g) \frac{\delta X(g)}{\delta g_s(y)} \,. \tag{C1}$$

Combining this result with Eq. (9.22), we find

$$\frac{\delta T_{ks'}(g)}{\delta g_s(y)} = \frac{\delta_{ss'}}{M_{0s}(g)}\left[y^{k+1} - y T_{ks}(g)\right] e^{-X(g)y} - \left[T_{k+1,s'}(g) - T_{ks'}(g) T_{1s'}(g)\right] \frac{\delta X(g)}{\delta g_s(y)} \,. \tag{C2}$$

The variational derivative of the identity (9.45) with respect to $\delta g_s(y)$ yields

$$\sum_{s'} \frac{\delta T_{1s'}(g)}{\delta g_s(y)} = 0\,. \tag{C3}$$

We insert Eq. (C2) into Eq. (C3):

$$\sum_{s'} \left\{ \frac{\delta_{ss'}}{M_{0s}(g)} \left[y^2 - yT_{1s}(g)\right] e^{-X(g)y} - \left\{T_{2s'}(g) - [T_{1s'}(g)]^2\right\} \frac{\delta X(g)}{\delta g_s(y)} \right\} = 0\,. \tag{C4}$$

Then

$$\frac{\delta X(g)}{\delta g_s(y)} = \frac{N(g)}{M_{0s}(g)} \left[y - T_{1s}(g)\right] y e^{-X(g)y}\,, \tag{C5}$$

where

$$N(g) = \left\{ \sum_{s'} \left\{T_{2s'}(g) - [T_{1s'}(g)]^2\right\} \right\}^{-1}\,. \tag{C6}$$

2. Identities for the action of the evolution kernel K

a. Action of K on $\delta W_0/\delta g \otimes \delta X/\delta g$

Now we want to prove the following identity

$$(g \otimes g) \cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta X(g)}{\delta g} \right] = 0\,. \tag{C7}$$

According to Eqs. (8.10) and (9.21) we have

$$\frac{\delta W_0(g)}{\delta g_s(y)} = \frac{y e^{-X(g)y}}{2M_{0s}(g)}\,. \tag{C8}$$

Using Eqs. (C5) and (C8), we find

$$\begin{aligned}
&(g \otimes g) \cdot K \cdot \left[\frac{\delta W_0(g)}{\delta g} \otimes \frac{\delta X(g)}{\delta g} \right] \\
&= \sum_{s_1 s_2} (g_{s_1} \otimes g_{s_2}) \cdot \tilde{K}^{s_1 s_2} \left\{ \left[\frac{y_1 e^{-X(g)y_1}}{2M_{0s_1}(g)} \right] \left\{ y_2 e^{-X(g)y_2} \frac{N(g)}{M_{0s_2}(g)} \left[y_2 - T_{1s_2}(g)\right] \right\} \right\} \\
&= \frac{N(g)}{2} \sum_{s_1 s_2} \frac{1}{M_{0s_1}(g) M_{0s_2}(g)} \left\{ (g_{s_1} \otimes g_{s_2}) \cdot \tilde{K}^{s_1 s_2} \left\{ \left[y_1 y_2 e^{-X(g)(y_1+y_2)} \right] \left[y_2 - T_{1s_2}(g)\right] \right\} \right\}\,.
\end{aligned} \tag{C9}$$

In order to compute the action of the operator K we use Eq. (B17)

$$\begin{aligned}
&\tilde{K}^{s_1 s_2} \left\{ \left[y_1 y_2 e^{-X(g)(y_1+y_2)} \right] \left[y_2 - T_{1s_2}(g)\right] \right\} \\
&= \tilde{K}^{s_1 s_2} \left\{ \left[y_1 y_2 e^{-X(g)(y_1+y_2)} \right] \left[-T_{1s_2}(g) + \frac{1}{2}(y_1 + y_2) + \frac{1}{2}(y_2 - y_1) \right] \right\} \\
&= \frac{1}{2} \left\{ \left[y_1 y_2 e^{-X(g)(y_1+y_2)} \right] \left\{ \gamma_0^{s_1 s_2} \left[-T_{1s_2}(g) + \frac{1}{2}(y_1 + y_2) \right] + \frac{1}{2} \gamma_1^{s_1 s_2} (y_2 - y_1) \right\} \right\}\,.
\end{aligned} \tag{C10}$$

Inserting this into Eq. (C9), we obtain

$$
\begin{aligned}
&(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta X(g)}{\delta g}\right]\\
&=\frac{N(g)}{4}\sum_{s_1s_2}\left\{\gamma_0^{s_1s_2}\left\{-T_{1s_2}(g)+\frac{1}{2}\left[T_{1s_1}(g)+T_{1s_2}(g)\right]\right\}-\frac{1}{2}\gamma_1^{s_1s_2}\left[T_{1s_1}(g)-T_{1s_2}(g)\right]\right\}\\
&=\frac{N(g)}{8}\sum_{s_1s_2}(\gamma_0^{s_1s_2}-\gamma_1^{s_1s_2})\left[T_{1s_1}(g)-T_{1s_2}(g)\right]
\end{aligned}
\tag{C11}
$$

Here

$$\gamma_0^{s_1s_2}-\gamma_1^{s_1s_2}=\gamma_0^{s_2s_1}-\gamma_1^{s_2s_1} \tag{C12}$$

is symmetric in s_1, s_2 whereas

$$T_{1s_1}(g)-T_{1s_2}(g) \tag{C13}$$

is antisymmetric. Therefore

$$\sum_{s_1s_2}(\gamma_0^{s_1s_2}-\gamma_1^{s_1s_2})\left[T_{1s_1}(g)-T_{1s_2}(g)\right]=0\,. \tag{C14}$$

Thus the RHS of Eq. (C11) vanishes. This proves the identity (C7).

b. *Action of K on $\delta W_0/\delta g\otimes\delta M_{0s}/\delta g$*

Taking $k=0$ in Eq. (C1) and using Eq. (C7), we find

$$
\begin{aligned}
&(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta M_{0s}(g)}{\delta g}\right]\\
&=\sum_{s_1s_2}\left\{(g_{s_1}\otimes g_{s_2})\cdot\tilde{K}^{s_1s_2}\left\{\frac{\delta W_0(g)}{\delta g_{s_1}(y_1)}\left[\delta_{ss_2}y_2e^{-X(g)y_2}\right]\right\}\right\}.
\end{aligned}
\tag{C15}
$$

Inserting Eq. (C8), we obtain

$$
\begin{aligned}
&(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta M_{0s}(g)}{\delta g}\right]\\
&=\sum_{s_1}\frac{1}{2M_{0s_1}}(g_{s_1}\otimes g_s)\cdot\tilde{K}^{s_1s}\left[y_1y_2e^{-X(g)(y_1+y_2)}\right].
\end{aligned}
\tag{C16}
$$

Here according to Eq. (8.1)

$$\tilde{K}^{s_1s}\left[y_1y_2e^{-X(g)(y_1+y_2)}\right]=\frac{1}{2}\delta_{s_1s}\left[y_1y_2e^{-X(g)(y_1+y_2)}\right]. \tag{C17}$$

Therefore

$$(g_{s_1}\otimes g_s)\cdot\tilde{K}^{s_1s}\left[y_1y_2e^{-X(g)(y_1+y_2)}\right]=\frac{1}{2}\delta_{s_1s}M_{0s_1}M_{0s}\,. \tag{C18}$$

Thus

$$(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta M_{0s}(g)}{\delta g}\right]=\frac{1}{4}M_{0s}(g)\,. \tag{C19}$$

As a consequence, we obtain the identity

$$(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta}{\delta g}\frac{M_{0+}(g)}{M_{0-}(g)}\right]=0\,. \tag{C20}$$

c. Action of K on $\delta W_0/\delta g \otimes \delta T_{ks}/\delta g$

Now we want to derive relation (9.26). According to Eqs. (C2) and (C7) we have

$$
\begin{aligned}
&(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta T_{ks}(g)}{\delta g}\right]\\
&=\sum_{s_1 s_2}(g_{s_1}\otimes g_{s_2})\cdot\tilde{K}^{s_1 s_2}\left\{\frac{\delta W_0(g)}{\delta g_{s_1}(y_1)}\left\{\frac{\delta_{ss_2}}{M_{0s}(g)}\left[y_2^{k+1}-y_2 T_{ks}(g)\right]e^{-X(g)y_2}\right\}\right\}.
\end{aligned}
\tag{C21}
$$

Inserting Eq. (C8), we find

$$
\begin{aligned}
&(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta T_{ks}(g)}{\delta g}\right]\\
&=\sum_{s_1}\frac{1}{2M_{0s_1}(g)M_{0s}(g)}(g_{s_1}\otimes g_s)\cdot\tilde{K}^{s_1 s}\left\{\left[y_2^k-T_{ks}(g)\right]y_1 y_2 e^{-X(g)(y_1+y_2)}\right\}.
\end{aligned}
\tag{C22}
$$

Let us decompose u^k in Gegenbauer polynomials $C_n^{3/2}(1-2u)$.

$$
u^k=\sum_{n=0}^{k}\alpha_{nk}C_n^{3/2}(1-2u)\ .
\tag{C23}
$$

Taking here

$$
u=\frac{y_2}{y_1+y_2}\,,
\tag{C24}
$$

we find

$$
y_2^k=\sum_{n=0}^{k}\alpha_{nk}(y_1+y_2)^k C_n^{3/2}\left(\frac{y_1-y_2}{y_1+y_2}\right).
\tag{C25}
$$

In order to compute the action of the operator K in Eq. (C22) we use Eqs. (B10), (B17), and (C25):

$$
\begin{aligned}
&\tilde{K}^{s_1 s}\left\{y_2^k\left[(y_1 y_2)\,e^{-X(g)(y_1+y_2)}\right]\right\}\\
&=\frac{1}{2}\left[(y_1 y_2)\,e^{-X(g)(y_1+y_2)}\right]\sum_{n=0}^{k}\alpha_{nk}\gamma_n^{s_1 s}(y_1+y_2)^k C_n^{3/2}\left(\frac{y_1-y_2}{y_1+y_2}\right).
\end{aligned}
\tag{C26}
$$

Here we deal with a homogeneous polynomial in y_1, y_2:

$$
(y_1+y_2)^k\sum_{n=0}^{k}\alpha_{nk}\gamma_n^{s_1 s_2}C_n^{3/2}\left(\frac{y_1-y_2}{y_1+y_2}\right)=\sum_{m=0}^{k}B_{km}^{s_1 s_2}y_1^m y_2^{k-m}\,.
\tag{C27}
$$

We consider this equation as a definition of coefficients $B_{km}^{s_1 s_2}$. Now we have

$$
\tilde{K}^{s_1 s}\left\{y_2^k\left[(y_1 y_2)\,e^{-X(g)(y_1+y_2)}\right]\right\}=\frac{1}{2}\left[(y_1 y_2)\,e^{-X(g)(y_1+y_2)}\right]\sum_{m=0}^{k}B_{km}^{s_1 s}y_1^m y_2^{k-m}\,.
\tag{C28}
$$

According to Eq. (8.1)

$$
\tilde{K}^{s_1 s}\left[(y_1 y_2)\,e^{-X(g)(y_1+y_2)}\right]=\frac{1}{2}\delta_{s_1 s}\left[(y_1 y_2)\,e^{-X(g)(y_1+y_2)}\right].
\tag{C29}
$$

Now we take the difference of Eqs. (C28) and (C29)

$$
\begin{aligned}
&\tilde{K}^{s_1 s}\left\{\left[y_2^k-T_{ks}(g)\right]\left[(y_1 y_2)\,e^{-X(g)(y_1+y_2)}\right]\right\}\\
&=\frac{1}{2}\left[(y_1 y_2)\,e^{-X(g)(y_1+y_2)}\right]\left[\sum_{m=0}^{k}B_{km}^{s_1 s}y_1^m y_2^{k-m}-\delta_{s_1 s}T_{ks}(g)\right].
\end{aligned}
\tag{C30}
$$

We insert this expression into Eq. (C22)

$$\begin{aligned}
&(g\otimes g)\cdot K\cdot\left[\frac{\delta W_0(g)}{\delta g}\otimes\frac{\delta T_{ks}(g)}{\delta g}\right]\\
&=\sum_{s_1}\frac{1}{4M_{0s_1}(g)M_{0s}(g)}(g_{s_1}\otimes g_s)\cdot\left\{\left[(y_1y_2)\,e^{-X(g)(y_1+y_2)}\right]\left[\sum_{m=0}^{k}B_{km}^{s_1s}y_1^my_2^{k-m}-\delta_{s_1s}T_{ks}(g)\right]\right\}\\
&=\sum_{s_1}\frac{1}{4M_{0s_1}(g)M_{0s}(g)}\left[\sum_{m=0}^{k}B_{km}^{s_1s}M_{ms_1}(g)M_{k-m,s}(g)-\delta_{s_1s}T_{ks}(g)M_{0s_1}(g)M_{0s}(g)\right]\\
&=\frac{1}{4}\left[-T_{ks}(g)+\sum_{s_1}\sum_{n=0}^{k}B_{kn}^{s_1s}T_{ns_1}(g)T_{k-n,s}(g)\right]. \qquad \text{(C31)}
\end{aligned}$$

Thus relation (9.26) is proved.

3. Coefficients α_{nk} and $B_{kn}^{s_1s_2}$

a. *Properties of* α_{nk}

Coefficients α_{nk} are defined by Eq. (C23) which can be rewritten in the form

$$\left(\frac{1-z}{2}\right)^k=\sum_{n=0}^{k}\alpha_{nk}C_n^{3/2}\left(z\right). \qquad \text{(C32)}$$

Using the orthogonality of Gegenbauer polynomials (B24), we find

$$\alpha_{nk}=\frac{1}{h_n}\int_{-1}^{1}dz(1-z^2)\left(\frac{1-z}{2}\right)^k C_n^{3/2}(z)\,. \qquad \text{(C33)}$$

b. *Properties of* $B_{kn}^{s_1s_2}$

Coefficients $B_{kn}^{s_1s_2}$ are defined by Eq. (C27). According to Eq. (B14) the anomalous dimensions $\gamma_n^{s_1s_2}$ depend only on relative helicities:

$$\gamma_n^{ss}\equiv\gamma_n^{+}\,,\quad \gamma_n^{s,-s}=\gamma_n^{-}\,. \qquad \text{(C34)}$$

The same holds for the coefficients $B_{kn}^{s_1s_2}$ (C27) so that we can introduce $B_{kn}^{\pm}$:

$$B_{kn}^{ss}=B_{kn}^{+}\,,\quad B_{kn}^{s,-s}=B_{kn}^{-}\,. \qquad \text{(C35)}$$

In Sec. IX C we need the values for B_{kk}^{ε} and B_{k0}^{ε},

$$B_{kk}^{\varepsilon}=-\frac{2\delta_{\varepsilon,-}}{(k+1)(k+2)}-\frac{2}{k(k+1)}\quad(k>0)\,, \qquad \text{(C36)}$$

$$B_{k0}^{\varepsilon}=1-\frac{2}{k+2}\delta_{\varepsilon,-}+2\sum_{j=2}^{k+1}\frac{1}{j}\quad(k\geq 0)\,, \qquad \text{(C37)}$$

which will be computed below. At $k=0$ we have

$$B_{00}^{\varepsilon}=\delta_{\varepsilon,+}\,. \qquad \text{(C38)}$$

In order to derive relations (C36) and (C37) we rewrite Eq. (C27) in the form

$$(y_1+y_2)^k\sum_{n=0}^{k}\alpha_{nk}\gamma_n^{\varepsilon}C_n^{3/2}\left(\frac{y_1-y_2}{y_1+y_2}\right)=\sum_{m=0}^{k}B_{km}^{\varepsilon}y_1^my_2^{k-m}\,. \qquad \text{(C39)}$$

Inserting Eq. (C33), we find

$$(y_1+y_2)^k \sum_{n=0}^{k} \frac{1}{h_n}\gamma_n^\varepsilon C_n^{3/2}\left(\frac{y_1-y_2}{y_1+y_2}\right) \times \int_{-1}^{1} dz(1-z^2)\left(\frac{1-z}{2}\right)^k C_n^{3/2}(z) = \sum_{m=0}^{k} B_{km}^\varepsilon y_1^m y_2^{k-m}\,. \tag{C40}$$

The summation over n on the LHS can be extended to infinity since $C_n^{3/2}(z)$ is orthogonal to polynomials of degree smaller than n:

$$(y_1+y_2)^k \int_{-1}^{1} dz \left(\frac{1-z}{2}\right)^k (1-z^2) \times \left[\sum_{n=0}^{\infty} \frac{1}{h_n}\gamma_n^\varepsilon C_n^{3/2}\left(\frac{y_1-y_2}{y_1+y_2}\right) C_n^{3/2}(z)\right] = \sum_{m=0}^{k} B_{km}^\varepsilon y_1^m y_2^{k-m}\,. \tag{C41}$$

Changing the integration variable

$$z = 2x-1\,, \tag{C42}$$

we obtain

$$8(y_1+y_2)^k \int_{0}^{1} dx\, x\,(1-x)^{k+1} \times \left[\sum_{n=0}^{\infty} \frac{1}{h_n}\gamma_n^\varepsilon C_n^{3/2}\left(\frac{y_1-y_2}{y_1+y_2}\right) C_n^{3/2}(2x-1)\right] = \sum_{m=0}^{k} B_{km}^\varepsilon y_1^m y_2^{k-m}\,. \tag{C43}$$

Taking $\phi(x) = (1-x)^k$ in identity (B25) and inserting the result into Eq. (C43), we obtain

$$(y_1+y_2)^k \left\{(1-y)^k - 2\int_0^1 dx \left\{\left[\delta_{\varepsilon,-}(1-x)^k + \frac{(1-x)^k-(1-y)^k}{|x-y|}\right]_{y=\frac{y_1}{y_1+y_2}} \times \left[\frac{x}{y}\theta(y-x) + \frac{1-x}{1-y}\theta(x-y)\right]\right\}\right\} = \sum_{m=0}^{k} B_{km}^\varepsilon y_1^m y_2^{k-m}\,. \tag{C44}$$

Setting $y_1 = 0$ here, we find

$$\begin{aligned} B_{k0}^\varepsilon &= 1 - 2\int_0^1 dx \left[\delta_{\varepsilon,-}(1-x)^k + \frac{(1-x)^k-1}{x}\right](1-x) \\ &= 1 - \frac{2}{k+2}\delta_{\varepsilon,-} + 2\sum_{j=2}^{k+1}\frac{1}{j}\,. \end{aligned} \tag{C45}$$

Similarly taking $y_2 = 0$, we obtain

$$B_{kk}^\varepsilon = -2\int_0^1 dx \left[\delta_{\varepsilon,-}(1-x)^k + (1-x)^{k-1}\right] x = -\frac{2\delta_{\varepsilon,-}}{(k+1)(k+2)} - \frac{2}{k(k+1)}\,. \tag{C46}$$

Thus relations (C36) and (C37) are derived.

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